

# BF systems on graph cobordisms as topological cosmology

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## Abstract

A cosmological model connecting the evolution of universe with a sequence of topology changes described by a collection of specific graph cobordisms, is constructed. It is shown that an adequate topological field theory (of BF-type) can be put into relation to each graph cobordism. The explicit expressions for transition amplitudes (partition functions) are written in these BF-models and it is shown that the basic topological invariants of the graph cobordisms (intersection matrices) play the rôle of coupling constants between the formal analogues of electric and magnetic fluxes quantized à la Dirac, but with the use of Poicaré–Lefschetz duality. For a specific graph cobordism, the diagonal elements and eigenvalues of the intersection matrix reproduce the hierarchy of dimensionless low-energy coupling constants of the fundamental interactions acting in the real universe.

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# 1 Introduction

The main hypothesis we advance in this paper is that the spacetime topology determines (*via* an Abelian topological BF-type field theory) the number and hierarchy of coupling constants of the fundamental (pre-)interactions which are adequate to the topological structure of the real universe. Thus we continue to develop the principal idea of [1] that the primary values of coupling constants (which correspond to a vacuum without any local excitations) are topological invariants of four-dimensional spacetime manifold. The topological field theories [2, 3, 4, 5] are in fact exercises in calculation of topological invariants (the “theory of nothing”) [6] accompanied by certain physical interpretations. In particular, in our  $U(1)$  gauge BF-model (in the vein of works on global aspects of electric-magnetic duality [7, 8, 9, 10, 11, 12, 13]) a determination of the complete system of the gauge classes of BF system solutions (phase space) is equivalent to specification of topology of spacetime  $M$  because of the existence of isomorphisms of type

$$\text{Princ}(M) \stackrel{c}{\cong} H^2(M, \mathbb{Z})$$

between the group of principal  $U(1)$ -bundles over  $M$  and the group of cohomology classes of 2-cycles of  $M$  (see the formulae (2.10) – (2.13)) and since the groups of cohomological classes of cocycles with dimensions 1 and 3 are trivial for the set of four-dimensional manifolds under consideration, namely graph cobordisms, which are intensively studied by mathematicians in recent years [14, 15, 16, 17].

The problem is set as in self-consistent cosmological model building: to construct a spacetime manifold which admits fields configurations with non-zero fluxes through the collection of homologically non-trivial 2-cycles [7], *i.e.* closed surfaces that do not correspond to the boundary of a three-dimensional submanifold in  $M$ . A generalization of Dirac’s quantization conditions [7, 8, 10, 11, 13] (formulated on the basis of the Poincaré–Lefschetz duality) implies that these fluxes must be quantized. We show that the coupling constant matrix describing interactions between the quantized fluxes, coincides up to a scale factor (which also is found to be a topological invariant) with the basic topological invariant of spacetime  $M$ , *i.e.* with its intersection matrix  $Q$ . For the specially constructed graph manifold  $M_D^+(0)$  (see section 4), diagonal elements of the intersection matrix  $Q^+(0)$  reproduce rather exactly the hierarchy of dimensionless low-energy coupling (DLEC)

constants of the universal physical interactions acting in our universe. The other graph cobordisms  $M_D^+(t)$ ,  $t = -1, -2, -3, -4$ , constructed in the same section, probably describe earlier stages of cosmological evolution, which are distinguished from each other rather by the spacetime topology than by gauge groups of physical fields (the gauge group we use is  $U(1)$ ). Thus our model demonstrates that topological invariants may codify information about the physical interactions, which can be naturally introduced on a non-trivial topological space forming the spacetime background of the self-consistent cosmological model which involves qualitatively different evolutionary phases connected one to another by topology changes.

The relations between the topology fluctuations and the problem of fixing fundamental coupling constants are discussed widely, see for example [18, 19]. In particular, the solution of the cosmological-constant problem in terms of spontaneous topology changes is treated in [20, 21, 22, 23]. We propose a fairly different approach to these old problems by means of using cobordisms of sufficiently specific type to model the spacetime manifolds.

We remind that if the boundary of a compact four-dimensional topological space  $M$ ,  $\partial M$ , is a disjoint sum of two three-dimensional topological spaces  $\Sigma_{\text{in}}$  and  $\Sigma_{\text{out}}$ , the 3-tuple  $(M, \Sigma_{\text{in}}, \Sigma_{\text{out}})$  is called a cobordism [24]. Both  $\Sigma_{\text{in}}$  and  $\Sigma_{\text{out}}$  are often merely implied, thus the very topological space  $M$  is called a cobordism.

The paper is organized as follows. The section 2 contains the basic mathematical concepts which are used to construct a topological field theory on graph cobordisms. In the subsection 2.1 we reproduce the definitions of splicing and plumbing operations which are necessary to glue graph cobordisms according to algorithms codified by means of splice diagrams. These concepts are unusual for physicists, but are well developed by topologists [31, 26, 25]. The subsection 2.2 contains a review of the basic notions connected with the principal  $U(1)$ -bundles on four manifolds with nonempty boundaries in the style of [13]. Here we also discuss the non-trivial (but rather simple) (co-) homological properties of graph cobordisms. The following subsection (2.3) is dedicated to a brief survey of Poincaré–Lefschetz duality [27, 25] which we use instead of the common Hodge duality to define the formal analogues of electric and magnetic fluxes in our version of BF-theory. Moreover we remind the definition of the main topological invariant of four-dimensional manifolds, namely intersection form [31, 26] which serves for the determination of coupling constant matrices characterizing interactions of these “electric” and “magnetic” fluxes.

In the section 3 we construct the simple version of the Abelian BF-theory on graph cobordisms, also known as plumbed  $V$ -cobordisms [15]. In analogy with the electrodynamics with theta term [7, 8, 13], the “electric” and “magnetic” fluxes are defined as linear combinations of the first Chern classes of graph cobordism  $M$  and its boundary  $\partial M$ . Then the transition amplitudes (partition functions) are expressed as functionals of these fluxes, intersection matrices and the BF scale factor  $\lambda$ . These transition amplitudes are topological invariants and represent something resembling to the theta function. They also support a certain form of strong-weak coupling duality.

In the section 4 we present numerical calculations of intersection matrices for a specific sequence of graph cobordisms which can be interpreted as a series of the topology changes leading to a certain state of the universe. This state can be identified with the contemporary one by means of the elementary interactions between “electric” fluxes, since the coupling constants hierarchy of these fluxes reproduces the hierarchy of the fundamental interactions in the real universe. At the end of this section we give interpretation and discussion of the obtained results.

The standard notations  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are used for the sets of integer, real and complex numbers, respectively.

## 2 Mathematical concepts

As this was said in the Introduction, we build here a rather simple topological gauge (BF-) model on sufficiently complicated topological spaces belonging to the class of graph cobordisms imitating tunnelling topological changes. These four-dimensional smooth manifolds with the Euclidean signature possess non-trivial (co-)homological characteristics. This leads to a specific generalization of the Dirac quantization conditions [7, 8, 10] and enables us to explicitly express transition amplitudes (partition functions) in terms of the topological invariants (intersection forms) of the graph cobordisms. The boundary components of graph cobordisms are disjoint sums of lens spaces and  $\mathbb{Z}$ -homology spheres. First we give some necessary definitions following the works of Saveliev [25, 15], see also [26].

## 2.1 $\mathbb{Z}$ -homology spheres, splicing, plumbing and graph cobordisms

Let  $a_1, a_2, a_3$  be pairwise relatively prime positive numbers. The Brieskorn homology sphere (Bh-sphere)  $\Sigma(\underline{a}) := \Sigma(a_1, a_2, a_3)$  is defined as the link of (Brieskorn) singularity

$$\Sigma(\underline{a}) := \Sigma(a_1, a_2, a_3) := \{z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap S^5 \quad (2.1)$$

where  $z_i \in \mathbb{C}_i$ , and  $S^5$  is the unit five-dimensional sphere  $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ . The singular complex algebraic surface  $z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0$  has the canonical orientation which induces the canonical orientation of the link  $\Sigma(\underline{a})$ . If any of  $a_i$  is equal to 1, the manifold  $\Sigma(\underline{a})$  is homeomorphic to the ordinary  $S^3$ . Bh-spheres belong to the class of Seifert fibered homology (Sfh-) spheres [28]. On this manifold, there exists a unique Seifert fibration which has unnormalized Seifert invariants [29]  $(a_i, b_i)$  subject to  $e(\Sigma(\underline{a})) = \sum_{i=1}^3 b_i/a_i = 1/a$ , where  $a = a_1 a_2 a_3$  and  $e(\Sigma(\underline{a}))$  is its Euler number (the well known topological invariant of a Bh-sphere). This Seifert fibration is defined by the  $S^1$ -action which reads  $t(z_1, z_2, z_3) = (t^{\sigma_1} z_1, t^{\sigma_2} z_2, t^{\sigma_3} z_3)$ , where  $t \in S^1$ , and  $\sigma_i = a/a_i$ . This action is fixed-point-free. The only points of  $\Sigma(a_1, a_2, a_3)$  which have non-trivial isotropy group  $\mathbb{Z}_{a_i}$  are those with one coordinate  $z_i$  equal to 0 ( $i = 1, 2, 3$ ). The fiber through such a point is called an exceptional (singular) fiber of degree  $a_i$ . All other fibers are called regular (non-singular). In general, Sfh-spheres  $\Sigma(a_1, \dots, a_n)$  have  $n$  different exceptional fibers and represent special cases of  $\mathbb{Z}$ -homology spheres [26].

By a  $\mathbb{Z}$ -homology sphere we mean a closed three-manifold  $\Sigma$  such that all homology groups of  $\Sigma$  with integer coefficients are isomorphic to homology groups of the ordinary three-sphere  $S^3$  over  $\mathbb{Z}$ . All  $\mathbb{Z}$ -homology spheres used in this paper can be obtained from Bh-spheres by the splicing operation. This operation is defined for any Sfh-sphere as follows: First we define [26] a Seifert link as a pair  $(\Sigma, S) = (\Sigma, S_1 \cup \dots \cup S_m)$  consisting of oriented  $\mathbb{Z}$ -homology sphere  $\Sigma$  and a collection  $S$  of Seifert fibers (exceptional or regular)  $S_1, \dots, S_m$  in  $\Sigma$ . Note that the links  $(S^3, S)$  where  $S^3$  is an ordinary three-sphere, are also allowed. Let  $(\Sigma, S)$  and  $(\Sigma', S')$  be links and choose components  $S_i \in S$  and  $S'_j \in S'$ . Let also  $N(S_i)$  and  $N(S'_j)$  be their tubular neighbourhoods, while  $m, l \subset \partial N(S_i)$  and  $m', l' \subset \partial N(S'_j)$  be standard meridians and longitudes. The manifold  $\Sigma'' = (\Sigma \setminus \text{int} N(S_i)) \cup (\Sigma' \setminus \text{int} N(S'_j))$  obtained by pasting along the torus boundaries by matching  $m$  to  $l'$  and  $m'$  to  $l$ , is a  $\mathbb{Z}$ -homology sphere. The link  $(\Sigma'', (S \setminus S_i) \cup (S' \setminus S'_j))$  is called the

splice (splicing) of  $(\Sigma, S)$  and  $(\Sigma', S')$  along  $S_i$  and  $S'_j$ . We shall use the standard notation  $\Sigma'' = \Sigma \overline{S_i S'_j} \Sigma'$  or simply  $\Sigma'' = \Sigma - \Sigma'$ . Any link which can be obtained from a finite number of Seifert links by splicing is called a graph link. Empty graph links are precisely the (graph)  $\mathbb{Z}$ -homology spheres. All graph links are classified as in [26] by their splice diagrams.

A splice diagram  $\Delta$  is a finite tree graph with vertices of three types. Vertices with at least three adjacent edges are called *nodes*. Each node with  $n$  adjacent edges corresponds to a Sfh-sphere  $\Sigma(a_1, \dots, a_n)$ . The edges adjacent to a node correspond to exceptional fibers and are weighted by  $a_1, \dots, a_n$ , respectively. The very node carries a sign plus if  $\Sigma(a_1, \dots, a_n)$  is oriented as a link of singularity of the type (2.1) and minus otherwise. Two other types of vertices have only one adjacent edge and are called either *leaves* or *arrowheads*. The former ones represent singular fibers in a Sfh-sphere, while the latter ones, components of a graph link. We shall use splice diagrams  $\Delta_r$  with nodes of valence  $n = 3$  only, see figure 1. This type of splice diagrams corresponds to splicing of  $r$  Bh-spheres. We just consider the splice diagrams with pairwise coprime positive weights around each node. In this case the concepts of integer ( $\mathbb{Z}$ -) homology spheres and graph homology spheres are equivalent.

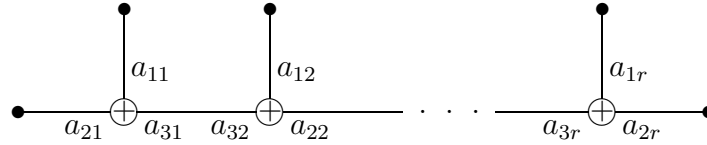


Figure 1: A splice diagram  $\Delta_r$ .

In general, graph homology spheres can be conveniently described by plumbing. Plumbing graphs are required for introduction of four-dimensional manifolds with graph homology spheres as boundaries. The plumbing representation make it also possible to define intersection forms of these four-manifolds and to pass to the definition of graph cobordisms.

A *plumbing graph*  $\Gamma$  is a graph with no cycles (a finite tree) each of whose vertices  $v_i$  carries an integer weight  $e_i$ ,  $i = 1, \dots, r$ . To each vertex  $v_i$  a  $D^2$ -bundle  $Y(e_i)$  over  $S^2$  is associated, whose Euler class (self-intersection number of zero-section) is  $e_i$ . If the vertex  $v_i$  has  $d_i$  edges connected to it

on the graph  $\Gamma$ , choose  $d_i$  disjoint discs in the base  $S^2$  of  $Y(e_i)$  and call the disc bundle over the  $j$ th disc  $B_{ij} = (D_j^2 \times D^2)_i$ . When two vertices  $v_i$  and  $v_k$  are connected by an edge, the disc bundles  $B_{ij}$  and  $B_{kl}$  should be identified by exchanging the base and fiber coordinates [30]. This pasting operation is called *plumbing*, and the resulting smooth four-manifold  $P(\Gamma)$  is known as *graph manifold* (*plumbed four-manifold*). Its boundary  $\Sigma(\Gamma) = \partial P(\Gamma)$  is referred to as a *plumbed three-manifold*.

Since the homology group  $H_1(P(\Gamma), \mathbb{Z}) = 0$ , the unique non-trivial homology characteristic is  $H_2(P(\Gamma), \mathbb{Z})$  which has a natural basis (set of generators) represented by the zero-sections of the plumbed bundles. All these sections are embedded 2-spheres  $z_i$  where  $i = 1, \dots, r = \text{rank } H_2(P(\Gamma), \mathbb{Z})$ , and they can be oriented in such a way that the intersection (bilinear) form [25]

$$Q : H_2(P(\Gamma), \mathbb{Z}) \otimes H_2(P(\Gamma), \mathbb{Z}) \rightarrow \mathbb{Z} \quad (2.2)$$

will be represented by the  $r \times r$ -matrix  $Q(\Gamma) = (q_{ij})$  with the entries:  $q_{ij} = e_i$  if  $i = j$ ;  $q_{ij} = 1$  if the vertex  $v_i$  is connected to  $v_j$  by an edge; and  $q_{ij} = 0$  otherwise. The three-manifold  $\Sigma(\Gamma)$  is  $\mathbb{Z}$ -homology sphere iff the matrix  $Q(\Gamma)$  is unimodular, that is  $\det Q(\Gamma) = 1$ .

In order to construct a plumbing representation for a  $\mathbb{Z}$ -homology sphere given by a splice diagram  $\Delta$ , we need two things:

1. Plumbing graphs for the basic building blocks, *i.e.* Sfh-spheres (in our case, Bh-spheres).
2. A procedure to splice together plumbing graphs.

First, let  $\Sigma$  be a Sfh-sphere with unnormalized Seifert invariants  $(a_i, b_i)$ ,  $i = 1, \dots, n$ , and the splice diagram of figure 2. It can be obtained as a boundary of the graph manifold  $P(\Gamma)$  where  $\Gamma$  is a star graph shown in figure 3 [30]. The integer weights  $t_{ij}$  in this graph are found from continued fractions  $a_i/b_i = [t_{i1}, \dots, t_{im_i}]$ ; here

$$[t_1, \dots, t_k] = t_1 - \frac{1}{t_2 - \frac{1}{\dots - \frac{1}{t_k}}}.$$

Lens spaces represent a special case of Seifert fibered manifolds. Expanding  $-p/q = [t_1, \dots, t_n]$  into a continued fraction, we encounter  $L(p, q)$  as a

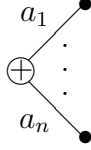


Figure 2: The splice diagram of an Sfh-sphere  $\Sigma$ .

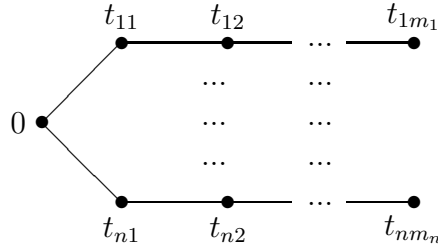


Figure 3: The star graph  $\Gamma$ .

boundary of the 4-manifold obtained by plumbing on the chain  $\Gamma_i^{\text{ch}}$  shown in figure 4.

Notice that this plumbing graph simultaneously represents the lens space  $L(p, q^*)$  with  $-p/q^* = [t_n, \dots, t_1]$  where  $qq^* = 1 \pmod p$ . This reflects the fact that  $L(p, q)$  and  $L(p, q^*)$  are homeomorphic. Moreover, to the continuous fraction  $a_i/b_i = [t_{i1}, \dots, t_{im_i}]$  there correspond both the subgraph  $\Gamma_i^{\text{ch}}$  of  $\Gamma$  shown in figure 4, and the lens space  $L(-a_i, b_i)$  (also called *leaf lens space*).

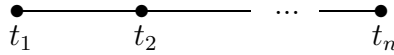


Figure 4: One-dimensional chain  $\Gamma_i^{\text{ch}}$ .

Next, we address the problem of splicing together plumbing graphs. It can be described as follows. Suppose that two graph links are represented by their plumbing diagrams  $\overline{\Gamma}$  and  $\overline{\Gamma'}$  (see figure 5) with arrows attached to



vertices  $e_n$  and  $e'_m$ , respectively. The corresponding plumbing diagram for a spliced link is shown in figure 6 where  $a = \det Q(\Gamma_0)/\det Q(\Gamma)$ , while  $\Gamma$  is the plumbing graph  $\bar{\Gamma}$  with the arrow deleted, and  $\Gamma_0$  is a portion of  $\Gamma$  obtained by removing the  $n$ th vertex weighted by  $e_n$  as well as all its adjacent edges. Another integer  $a'$  is similarly obtained from the graph  $\Gamma'$  (examples see in [26, 15]). The above description of splicing in terms of plumbing graphs makes it possible to treat splicing as an operation on the corresponding plumbed 4-manifold; moreover,  $\Sigma(\Gamma) = \partial P(\Gamma)$ .

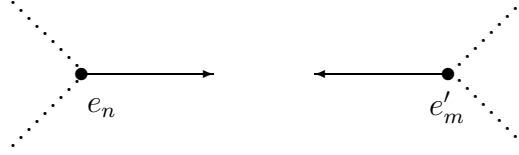


Figure 5: Plumbing diagrams  $\bar{\Gamma}$  and  $\bar{\Gamma}'$  prepared for splicing.

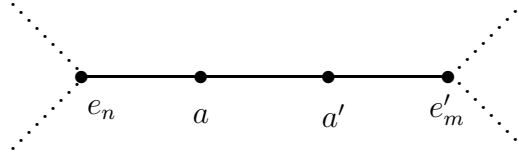


Figure 6: Plumbing diagram obtained by splicing  $\bar{\Gamma}$  and  $\bar{\Gamma}'$ .

The important problem in calculation with plumbed graphs is to identify the *extra lens space* arising between nodes in the course of splicing. See figure 7 where the lens space in question is decorated by oval. The resulting lens space  $L(p, q)$  ( $-p/q = [t_1, \dots, t_k]$ ) is characterized by the following parameters,

$$p = a_1 \cdots a_{n-1} \alpha_1 \cdots \alpha_{m-1} - a_n \alpha_m, \quad (2.3)$$

$$q = -a_1 \cdots a_{n-1} \alpha_1 \cdots \alpha_{m-1} \sum_{i=1}^{n-1} \frac{b_i}{a_i} - b_n \alpha_m. \quad (2.4)$$

This lens space depends only on the spliced Seifert links  $(\Sigma(a_1, \dots, a_n), S_n)$  and  $(\Sigma(\alpha_1, \dots, \alpha_m), S_m)$ , but does not depend on the rest of the splice diagram [15, 26].

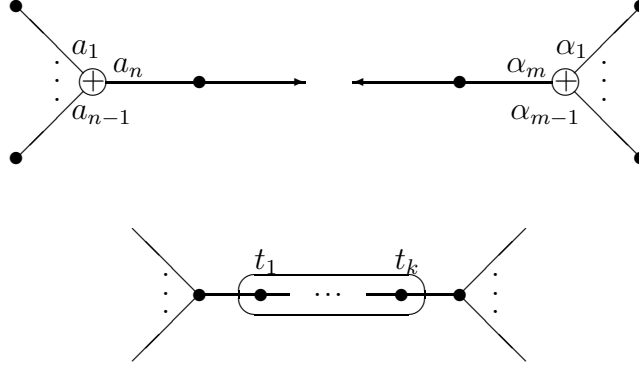


Figure 7: The extra lens space arising as a result of splicing.

If all extra lens spaces (which we also call “lens spaces between nodes”)  $L(p'_k, q'_k)$ ,  $k = 1, \dots, N_{\text{extra}}$ , are subject to the condition  $p'_k < 0$ , then  $\partial P(\Gamma) = \Sigma_{\text{alg}}$  is an *algebraic link* [26] which consequently bounds the graph manifold  $P(\Gamma)$  having a negative defined intersection form  $Q(\Gamma)$ .

Now we are ready to define topological spaces of most importance in this paper, the graph cobordisms also known as plumbed V-cobordisms [15] related to the decorated plumbed graphs. Let  $P(\Gamma)$  be a plumbed four-manifold corresponding to graph  $\Gamma$ , and  $\Gamma^{\text{ch}}$  be a chain in  $\Gamma$  of the form shown in Figure 4. Plumbing on  $\Gamma^{\text{ch}}$  yields a submanifold  $P(\Gamma^{\text{ch}})$  of  $P(\Gamma)$  whose boundary is a lens space  $L(p', q')$ . The closure of  $P(\Gamma) \setminus P(\Gamma^{\text{ch}})$  is a smooth compact 4-manifold with oriented boundary  $-L(p', q') \sqcup \partial P(\Gamma)$  where  $\sqcup$  denotes the disjoint sum operation. Starting with several chains  $\Gamma_s^{\text{ch}}$  ( $s = \overline{1, N}$ ) in  $\Gamma$  (where  $\overline{0, N}$  is the integer numbers interval from 0 to  $N$ ), one can introduce a cobordism  $P(\Gamma_D)$  between  $\Sigma(\Gamma) := \partial P(\Gamma)$  and the

disjoint union  $L = \bigsqcup_{s=1}^N L(p'_s, q'_s)$ , *i.e.*

$$\partial P(\Gamma_D) = \left( - \bigsqcup_{s=1}^N L(p'_s, q'_s) \right) \sqcup \Sigma(\Gamma). \quad (2.5)$$

Such a cobordism will be called *graph cobordism*. Here naturally appears the concept of a *decorated graph*  $\Gamma_D$  shown as an ordinary graph  $\Gamma$  but with ovals or circles, each enclosing exactly one chain  $\Gamma_s^{\text{ch}}$ .

*Observation 2.1.* The chains  $\{\Gamma_s^{\text{ch}}\}$  must be disjoint in the following sense: No two chains should have a common vertex, and no edges of  $\Gamma$  should have one endpoint on one chain and another, on any other chain.

*Observation 2.2.* Consider the algebraic case when all extra lens spaces  $L(p'_k, q'_k)$  are subject to the condition  $p'_k < 0$ , and there are decorated (by ovals)  $N_{\text{extra}}$  extra lens spaces and  $N_{\text{leaf}}$  leaf lens spaces  $L(-a_i, b_i) \equiv L(p'_i, q'_i)$ . In this case the cobordism  $P(\Gamma_D)$  has the boundary

$$\partial P(\Gamma_D) = \left( - \bigsqcup_{k=1}^{N_{\text{extra}}} L(p'_k, q'_k) \right) \left( - \bigsqcup_{i=1}^{N_{\text{leaf}}} L(p'_i, q'_i) \right) \bigsqcup \Sigma_{\text{alg}}, \quad (2.6)$$

and its intersection matrix  $Q(\Gamma_D)$  is negative defined. The series defining transition amplitudes (partition functions) in section 3 converge if the intersection matrix of the four-dimensional cobordism is positive defined. Hence the orientation of the spacetime cobordism describing a topological tunnelling should be inverse to that of  $P(\Gamma_D)$ . We define the graph cobordism as  $M_D = -P(\Gamma_D) = P(-\Gamma_D)$  where  $-\Gamma_D$  is the decorated graph  $\Gamma_D$  with all weights being sign-inverse. Thus

$$\partial M_D = \left( - \bigsqcup_{k=1}^{N_{\text{extra}}} L(|p'_k|, q'_k) \right) \left( - \bigsqcup_{i=1}^{N_{\text{leaf}}} L(a_i, b_i) \right) \bigsqcup \Sigma \quad (2.7)$$

where  $\Sigma = \Sigma(-\Gamma) = -\Sigma_{\text{alg}}$ , and it was taken into account that  $p'_k < 0$  and  $a_i = -p'_i > 0$ . Since  $N = N_{\text{extra}} + N_{\text{leaf}}$ , (2.7) now reads

$$\partial M_D = \left( - \bigsqcup_{s=1}^N L(|p'_s|, q'_s) \right) \bigsqcup \Sigma. \quad (2.8)$$

The resulting orientation of the four-manifold  $M_D$  coincides with that introduced by Hirzebruch in [31] for plumbed manifolds. The graph cobordisms  $M_D$  have positive defined intersection forms. In section 4 we build examples of such cobordisms as a spacetime basis for cosmological models.

## 2.2 Principal $U(1)$ -bundles, connections and cohomologies

Let  $M$  be a four-manifold with boundary  $\partial M$ . We denote by  $i_\partial : \partial M \rightarrow M$  the natural inclusion map and by  $d$ , the de Rham differential on  $M$ . By  $H^p(M, \mathbb{Z})$  we further denote the absolute  $p$ th cohomology group with integer coefficients, and by  $H^p(M, \partial M, \mathbb{Z})$ , the relative  $p$ th cohomology group modulo  $\partial M$ . We also denote by  $\Omega^p(M) = C^p(M, \mathbb{R})$  the space of  $p$ -forms on  $M$ , and by  $\Omega^p(M, \partial M) = C^p(M, \partial M, \mathbb{R})$  the space of relative  $p$ -forms on  $M$ . The subscript  $\mathbb{Z}$  is attached for corresponding subsets of  $p$ -forms with integer (relative) periods:

$$\int_\Sigma f \in \mathbb{Z} \quad (2.9)$$

where  $f \in \Omega_\mathbb{Z}^p(M)$  (or  $\Omega_\mathbb{Z}^p(M, \partial M)$ ), while  $\Sigma$  is (relative) closed  $p$ -dimensional surface (cycle).

The quantization of BF-theory and evaluation of the transition amplitude corresponding to the cobordism  $M$  involve summation over the topological classes of gauge fields. Mathematically, these classes can be identified with the isomorphism classes of principal  $U(1)$ -bundles.

Let  $\text{Princ}(M)$  be the group of principal  $U(1)$ -bundles over  $M$ . It is well known that there exists the isomorphism

$$\text{Princ}(M) \stackrel{c}{\cong} H^2(M, \mathbb{Z}) \quad (2.10)$$

which assigns to a bundle  $P$  its first Chern class  $c(P)$ , see, *e.g.*, [12, 13]. Since  $H^2(M, \mathbb{Z})$  is an Abelian group, the subset of elements of finite order is a subgroup,  $\text{Tor}H^2(M, \mathbb{Z})$ , called the torsion subgroup of  $H^2(M, \mathbb{Z})$ . The preimage by the map  $c$  of the torsion subgroup is the subgroup  $\text{Princ}_0(M)$  of  $\text{Princ}(M)$ , the elements of which are called flat principal  $U(1)$ -bundles, *i.e.*

$$\text{Princ}_0(M) \stackrel{c}{\cong} \text{Tor}H^2(M, \mathbb{Z}). \quad (2.11)$$

A relative principal  $U(1)$ -bundle  $(P, t)$  on  $M$  consists both of principal  $U(1)$ -bundle  $P$  on  $M$  such that its restriction  $i_\partial^*P$  on  $\partial M$  is trivial, and of trivialization  $t : i_\partial^*P \rightarrow \partial M \times U(1)$ . The relative principal  $U(1)$ -bundles form a group  $\text{Princ}(M, \partial M)$  which is isomorphic to the 2nd relative cohomology group,

$$\text{Princ}(M, \partial M) \stackrel{c_{\text{rel}}}{\cong} H^2(M, \partial M, \mathbb{Z}), \quad (2.12)$$

where the map  $c_{\text{rel}}$  assigns to a relative bundle  $(P, t)$  its relative first Chern class  $c(P, t) = c_{\text{rel}}(P)$ .

Of course, we can describe the group  $\text{Princ}(\partial M)$  of principal  $U(1)$ -bundles on  $\partial M$  in the same way as we did for the group  $\text{Princ}(M)$ . Thus the isomorphism (2.10) holds in the form

$$\text{Princ}(\partial M) \xrightarrow{c_{\partial}} H^2(\partial M, \mathbb{Z}). \quad (2.13)$$

The preimage of  $\text{Tor}H^2(\partial M, \mathbb{Z})$  relative to the map  $c_{\partial}$  is the subgroup  $\text{Princ}_0(\partial M)$  of flat principal  $U(1)$ -bundles on  $\partial M$ .

Consider now the problem of extendability of principal bundles on  $\partial M$  to  $M$  following Zucchini [13]. This is important since the gauge theory transition amplitude (partition function) involves a sum over the set of the bundles  $P \in \text{Princ}(M)$  such that their restrictions  $i_{\partial}^*P$  on the boundary  $\partial M$  coincide with a fixed bundle  $P_{\partial} \in \text{Princ}(\partial M)$ . Every bundle  $P \in \text{Princ}(M)$  yields by pull-back  $i_{\partial}^* : P \rightarrow P_{\partial}$  (induced by the natural inclusion  $i_{\partial} : \partial M \rightarrow M$ ) a bundle  $P_{\partial} \in \text{Princ}(\partial M)$ . But the converse is in general false: not every bundle  $P_{\partial}$  is a pull-back of some bundle  $P$ . When this does indeed happen, one says that  $P_{\partial}$  is extendable to  $M$ . We shall show that in the case of four-dimensional graph cobordisms (considered in this paper as a model of spacetime) any  $P_{\partial} \in \text{Princ}(\partial M)$  is extendable. To this end, consider the absolute/relative cohomology exact sequence:

$$\dots \rightarrow H^p(\partial M, \mathbb{Z}) \rightarrow H^{p+1}(M, \partial M, \mathbb{Z}) \rightarrow H^{p+1}(M, \mathbb{Z}) \rightarrow H^{p+1}(\partial M, \mathbb{Z}) \rightarrow \dots \quad (2.14)$$

We can now use the isomorphisms (2.10), (2.12) and (2.13) to draw the commutative diagram

$$\begin{array}{ccccccc} H^1(\partial M, \mathbb{Z}) & \rightarrow & H^2(M, \partial M, \mathbb{Z}) & \xrightarrow{j^*} & H^2(M, \mathbb{Z}) & \xrightarrow{i_{\partial}^*} & H^2(\partial M, \mathbb{Z}) \xrightarrow{\delta^*} H^3(M, \partial M, \mathbb{Z}) \\ & & c_{\text{rel}} \uparrow & & c \uparrow & & c_{\partial} \uparrow \\ & & \text{Princ}(M, \partial M) & \xrightarrow{j^*} & \text{Princ}(M) & \xrightarrow{i_{\partial}^*} & \text{Princ}(\partial M) \end{array} \quad (2.15)$$

in which the lines are exact and vertical maps are isomorphisms. In the case of graph cobordisms under consideration it is true that [14, 15]

$$H^3(M, \partial M, \mathbb{Z}) = 0, \quad (2.16)$$

$$H^1(\partial M, \mathbb{Z}) = 0. \quad (2.17)$$

Interpretation of the second line in (2.15) is quite simple: the mapping  $j^*$  associates with every relative bundle  $(P, t) \in \text{Princ}(M, \partial M)$  the underlying bundle  $P \in \text{Princ}(M)$ , and the mapping  $i_\partial^*$  associates with every bundle  $P$  its pull-back bundle  $P_\partial = i_\partial^* P \in \text{Princ}(\partial M)$ . This interpretation then applies to the first line too.

By the exactness of lines in (2.15) the bundle  $P_\partial \in \text{Princ}(\partial M)$  is the pull-back of the bundle  $P \in \text{Princ}(M)$  iff

$$\delta^*(c_\partial(P_\partial)) = 0. \quad (2.18)$$

Consequently, the obstruction to the extendability of  $P_\partial$  is a class of  $H^3(M, \partial M, \mathbb{Z})$ . Since (2.16) is true for all our graph cobordisms, every principal  $U(1)$ -bundles on  $\partial M$  are extendable onto  $M$ . Each  $P_\partial$  has several extensions to  $M$ . Again, by the exactness of (2.15), its extensions are parametrized by the group of relative bundles  $\text{Princ}(M, \partial M)$ . This parametrization is one-to-one due to (2.17) (the mapping  $j^*$  is injective).

## Connections

Let  $P \in \text{Princ}(M)$  be a principal  $U(1)$ -bundle. We denote by  $\text{Conn}(P)$  the affine space of connections of  $P$ . Let  $A \in \text{Conn}(P)$  be a connection on  $P$ . We can fix a trivializing cover  $\{U_\alpha\}$  of  $M$  ( $M = \cup_\alpha U_\alpha$ ) and assign to each open set  $U_\alpha$  a vector potential  $A^\alpha$ . The connection  $\{A^\alpha\}$  is a Čech 0-cochain with values in 1-forms  $A^\alpha = A_i^\alpha dx^i$  [33]. The curvature  $F_A$  of a connection  $A$  is defined by

$$F_A = dA. \quad (2.19)$$

This is a brief expression of the local relations  $F_A^\alpha = F_A|_{U_\alpha} = dA^\alpha$ . The gauge transformation properties of  $A$  [9]

$$A^\alpha - A^\beta = d\chi^{\alpha\beta} \quad (2.20)$$

ensure that  $F_A$  does not depend on the chosen local trivialization of  $P$ , *i.e.*  $F$  is gauge invariant,  $F^\alpha = F^\beta$  in  $U_\alpha \cap U_\beta$ , thus  $F_A \in \Omega^2(M)$  is a 2-form,  $F_A$  obviously being closed:

$$dF_A = 0. \quad (2.21)$$

## 2.3 Intersection forms and the Poincaré–Lefschetz duality

Now let  $M$  be a graph cobordism corresponding to a decorated graph  $\Gamma_D$  (see subsection 2.1 for  $M_D$ ; we suppress the subindex  $_D$  below in this section). Then for any elements  $f, f' \in H^2(M, \mathbb{Z})$  the rational intersection number  $\langle f, f' \rangle_{\mathbb{Q}}$  is defined as follows [14]: We start with the part

$$0 \rightarrow H^2(M, \partial M, \mathbb{Z}) \xrightarrow{j^*} H^2(M, \mathbb{Z}) \xrightarrow{i_{\partial}^*} H^2(\partial M, \mathbb{Z}) \rightarrow 0 \quad (2.22)$$

of the exact sequence (2.14). Since  $H^2(\partial M, \mathbb{Z})$  is a pure torsion (finite Abelian group), we see that for any  $f \in H^2(M, \mathbb{Z})$  there exists  $p \in \mathbb{Z}$  with  $i_{\partial}^*(pf) = 0$ , hence  $pf = j^*(b)$  for unique  $b \in H^2(M, \partial M, \mathbb{Z})$ . Then we put

$$\langle f, f' \rangle_{\mathbb{Q}} := \frac{1}{p} \langle b, f' \rangle_{\mathbb{Z}} \in \mathbb{Q}, \quad (2.23)$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{Z}}$  on the right-hand side is the usual integer intersection number well defined due to the Poincaré–Lefschetz duality [25]

$$H^2(M, \mathbb{Z}) \cong H_2(M, \partial M, \mathbb{Z}), \quad (2.24)$$

$$H^2(M, \partial M, \mathbb{Z}) \cong H_2(M, \mathbb{Z}). \quad (2.25)$$

The Poincaré–Lefschetz duality pairing (PL-pairing)

$$\langle \cdot, \cdot \rangle_{\mathbb{Z}} : H^2(M, \partial M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \rightarrow \mathbb{Z} \quad (2.26)$$

can be written in the de Rham representation as

$$\langle b, f' \rangle = \int_M b \wedge f' \in \mathbb{Z}. \quad (2.27)$$

This is true since  $\Lambda = H^2(M, \partial M, \mathbb{Z})$  and  $\Lambda^{\#} = H^2(M, \mathbb{Z})$  are the integer cohomology lattices in the de Rham cohomology space  $H^2(M, \mathbb{R})$ . Note that if  $H^2(M, \partial M, \mathbb{Z})$  and  $H^2(M, \mathbb{Z})$  had torsion, this inclusion would be impossible, but in the case of graph cobordisms these groups are finitely generated free Abelian ones [14, 16]. Moreover,

$$\text{rank } H^2(M, \partial M, \mathbb{Z}) = \text{rank } H^2(M, \mathbb{Z}) = \text{rank } H^2(M, \mathbb{R})$$

since  $H^2(\partial M, \mathbb{Z})$  is pure torsion [34].

From exactness of the sequence (2.22) it follows that  $H^2(M, \partial M, \mathbb{Z})$  is the subgroup of  $H^2(M, \mathbb{Z})$  (the mapping  $j^*$  is monomorphism). Since both groups are torsion-free, the group  $H^2(M, \mathbb{Z})$  can be represented as a homomorphism group

$$H^2(M, \mathbb{Z}) = \text{Hom}(H_2(M, \mathbb{Z}), \mathbb{Z}) \cong \text{Hom}(H^2(M, \partial M, \mathbb{Z}), \mathbb{Z}), \quad (2.28)$$

*i.e.* as the lattice  $\Lambda^\#$  dual to  $\Lambda = H^2(M, \partial M, \mathbb{Z})$  with respect to the scalar product (PL-pairing)

$$\Lambda^\# = H^2(M, \mathbb{Z}) := \{f \in H^2(M, \mathbb{R}) \mid \langle b, f \rangle \in \mathbb{Z}, \forall b \in \Lambda\} \quad (2.29)$$

where  $\langle b, f \rangle$  is defined in de Rhamian representation by (2.27). Note that  $\Lambda \subset \Lambda^\#$ , thus  $\Lambda$  is not unimodular, and it is possible to introduce the nontrivial discriminant group

$$T(\Lambda) := \Lambda^\# / \Lambda = H^2(M, \mathbb{Z}) / H^2(M, \partial M, \mathbb{Z}) = H^2(\partial M, \mathbb{Z}) \quad (2.30)$$

which is the finite Abelian group. (The last equality follows from exactness of the sequence (2.22)).

For our purposes there will be useful the following

*Proposition* [14]. Let  $M$  be a graph cobordism. If we choose a certain basis  $b_I$  of  $\Lambda = H^2(M, \partial M, \mathbb{Z})$  and the dual basis  $f^I$  of  $\Lambda^\# = H^2(M, \mathbb{Z})$  ( $I = 1, \dots, r = \text{rank } H^2(M, \mathbb{Z})$ ), dual in the sense that  $\langle b_I, f^J \rangle_{\mathbb{Z}} = \delta_I^J$ , then the integral intersection matrix

$$Q_{IJ} = \langle b_I, b_J \rangle_{\mathbb{Z}} \quad (2.31)$$

for  $H^2(M, \partial M, \mathbb{Z})$ , is inverse of the rational intersection matrix

$$Q^{IJ} = \langle f^I, f^J \rangle_{\mathbb{Q}}. \quad (2.32)$$

Note that order of the discriminant group (2.30) is [10]

$$|T(\Lambda)| = \det \langle b_I, b_J \rangle_{\mathbb{Z}} = \det Q_{IJ}. \quad (2.33)$$

Let us now calculate the discriminant group  $T(\Lambda)$  defined in (2.30). The exactness of the cohomology sequence (2.22) results in existence of such a basis  $\{f^I \mid I \in \overline{1, r}\}$  in the group  $H^2(M, \mathbb{Z})$  that  $i_\partial^*(f^I) = t^I$  are generators of  $H^2(\partial M, \mathbb{Z})$  [32]. Due to finiteness of the group  $T(\Lambda) = H^2(\partial M, \mathbb{Z})$  there exist minimal integers  $p(I) > 1$  such that  $p(I)t^I = 0$  (without summation



in  $I$ ). Since the mapping  $i_{\partial}^*$  is linear,  $i_{\partial}^*(p(I)f^I) = p(I)t^I = 0$ . Moreover, the monomorphism property of  $j^*$  in (2.22) yields existence of the unique element  $\tilde{f}_I \in H^2(M, \partial M, \mathbb{Z})$  such that  $j^*(\tilde{f}_I) = p(I)f^I$ . The class  $j^*(\tilde{f}_I)$  can be considered as an element  $\tilde{f}^I$  in the subgroup  $H^2(M, \partial M, \mathbb{Z})$  of  $H^2(M, \mathbb{Z})$ . Due to  $\text{rank } H^2(M, \partial M, \mathbb{Z}) = \text{rank } H^2(M, \mathbb{Z}) = r$  and to the minimality of the integers  $p(I)$ , the set of classes  $\{\tilde{f}^I\} = \{p(I)f^I\}$  forms the basis in  $H^2(M, \partial M, \mathbb{Z})$  [14]. The theorems 5.1.1 and 5.1.3 in [34] then lead to the conclusion that the discriminant group  $T(\Lambda)$  reads as

$$T(\Lambda) = \bigoplus_{I=1}^r \mathbb{Z}_{p(I)}. \quad (2.34)$$

Orders  $p(I)$  of cyclic groups  $\mathbb{Z}_{p(I)}$  can be calculated from the characteristics of the decorated graph  $\Gamma_D$  which determines topology of the cobordism  $M$ , *i.e.* from the topological invariants of this graph cobordism. To this end note that the number of elements in the discriminant group  $T(\Lambda)$  is

$$|T(\Lambda)| = p(1) \cdots p(r) = \det Q_{IJ} = |p'_1 \cdots p'_{2r+1}|. \quad (2.35)$$

The last equality follows from juxtaposition of the generators  $\{f^I \mid I \in \overline{1, r}\}$  of  $H^2(M, \mathbb{Z})$  to the vertices  $\{v_I \mid I \in \overline{1, r}\}$  of the graph  $\Gamma_D$  outside of all the decorated ovals [15, 17]. In this case

$$\det Q_{IJ} = |p'_1 \cdots p'_{2r+1}| \quad (2.36)$$

where  $p'_s$  are characteristics of decorated ovals (decorated linear chains) of the graph  $\Gamma_D$ , see (2.8) with  $N = 2r + 1$  and the first decorated graph  $\Gamma_D^r$  in figure 8.

Now consider sublattices  $\Lambda_{r-1}$  and  $\Lambda_{r-1}^\#$  (of the lattices  $\Lambda_r$  and  $\Lambda_r^\#$ ) generated by the subbases  $\{\tilde{f}^I \mid I \in \overline{1, r-1}\}$  and  $\{f^I \mid I \in \overline{1, r-1}\}$ , respectively. This singles out from the graph  $\Gamma_D \equiv \Gamma_D^r$ , the subgraph  $\Gamma_D^{r-1}$  which consists of the vertices  $\{v_I \mid I \in \overline{1, r-1}\}$  and decorated ovals adjusted to these vertices, *i.e.* decorated chains with characteristics  $p'_1, \dots, p'_{2(r-1)+1} = p'_{2r-1}$ , see figure 8. The discriminant group corresponding to this subgraph is

$$T(\Lambda_{r-1}) = \Lambda_{r-1}^\# / \Lambda_{r-1} = \bigoplus_{I=1}^{r-1} \mathbb{Z}_{p(I)} \quad (2.37)$$

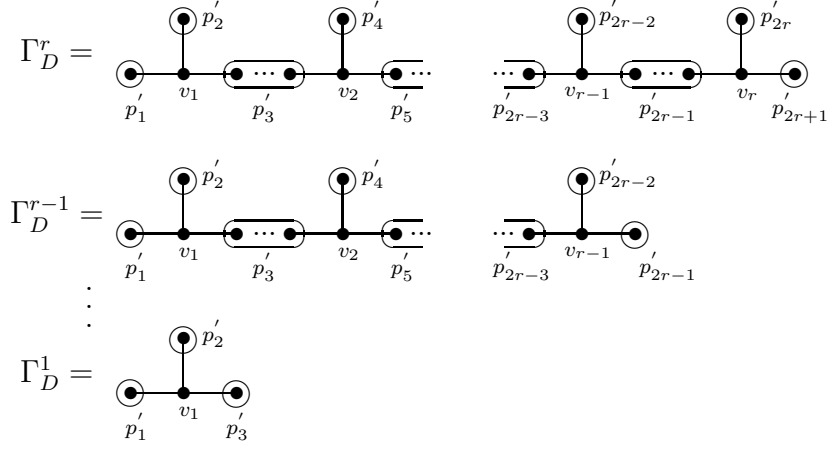


Figure 8: The decorated graphs  $\Gamma_D^I$ .

and possesses the order

$$|T(\Lambda_{r-1})| = p(1) \cdots p(r-1) = |p'_1 \cdots p'_{2r-1}|. \quad (2.38)$$

A comparison of (2.38) and (2.35) shows that

$$p(r) = |p'_{2r} p'_{2r+1}|. \quad (2.39)$$

After a finite number of steps, we encounter the following sequence of relations:

$$p(I) = |p'_{2I} p'_{2I+1}| \text{ for } 2 \leq I \leq r \text{ and finally } p(1) = |p'_1 p'_2 p'_3| \quad (2.40)$$

(the last expression is related to the final graph  $\Gamma_D^1$ ).

### 3 Topological gauge theory

#### 3.1 The classical Abelian BF-model

Let  $P \in \text{Princ}(M)$  be a principal  $U(1)$ -bundle on  $M$  and  $P_\partial := i_\partial^* P \in \text{Princ}(\partial M)$  be the induced principal  $U(1)$ -bundle on  $\partial M$ . The Abelian BF

gauge theory action is a functional of the connection  $A \in \text{Conn}(P)$  and of the auxiliary 2-cochain  $B \in C^2(M, \mathbb{R})$  (2-form in the de Rham representation) [4]

$$S = \frac{1}{2\pi} \int_M \left( B \wedge F - \frac{\lambda}{2} B \wedge B \right) \quad (3.1)$$

where  $F = dA$  is the curvature of the connection  $A$  (see subsection 2.2),  $\lambda$  being a scale factor analogous to the cosmological constant [5].

The dynamical equations following from (3.1) are quite simple,

$$F = \lambda B, \quad (3.2)$$

$$dB = 0, \quad (3.3)$$

if the normal boundary condition [13] on the variation of the connection  $\delta A$  is accepted,

$$i_{\partial}^*(\delta A) = 0. \quad (3.4)$$

Moreover the equation (3.3) is a consequence of (3.2) and the Bianchi identity  $dF = 0$ . The action (3.1) is actually invariant under very large gauge transformations

$$\delta A = w, \quad (3.5)$$

$$\delta B = \frac{1}{\lambda} dw \quad (3.6)$$

where  $w$  is an arbitrary 1-form. The gauge-inequivalent classical solutions of the BF-system (3.1) are thus characterized by a 2-cocycle  $F_{\text{cl}}$  modulo coboundary  $dw$  (*i.e.*  $F_{\text{cl}} \in H^2(M, \mathbb{R})$ ) and by a 2-cocycle  $B_{\text{cl}}$  modulo coboundary  $\frac{1}{\lambda} dw$  (*i.e.*  $B_{\text{cl}} \in H^2(M, \mathbb{R})$ ).

Quantization of the BF-theory involves a summation over topological classes of gauge fields. Formally, these classes may be identified with the isomorphism classes of principal  $U(1)$ -bundles  $\text{Princ}(M)$  and of relative principal  $U(1)$ -bundles  $\text{Princ}(M, \partial M)$  which were described in subsection 2.1 by means of isomorphisms (2.10) and (2.12). Thus it is natural to introduce the BF analogue of the generalized Dirac quantization conditions (*cf.* [7]) for a graph cobordism  $M$  as follows:

$$c_{\text{rel}}(P) = \frac{B_{\text{cl}}}{2\pi} \in \Lambda \cong H^2(M, \partial M, \mathbb{Z}) \subset H^2(M, \mathbb{R}), \quad (3.7)$$

$$c(P) = \frac{F_{\text{cl}}}{2\pi} \in \Lambda^{\#} \cong H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}). \quad (3.8)$$

### 3.2 Transition amplitudes

In an analogy with the partition function of Abelian gauge theory [7, 8] we construct the transition amplitude for purely topological gauge (BF-) theory over a graph cobordism  $M$ . As usual [13], this transition function involves both summation over the set of principal  $U(1)$  bundles  $P \in \text{Princ}(M) \cong H^2(M, \mathbb{Z})$  such that the principal bundle  $P_\partial = i_\partial^* P$  is fixed, and functional integration over bulk quantum fluctuations  $v$  of the connection on  $P$  which satisfies the ordinary normal boundary condition [35]

$$i_\partial^* v = 0. \quad (3.9)$$

We apply the customary classical-background-quantum-splitting method [36] which can be realized as follows. We write a general representative of cohomology class in  $H^2(M, \mathbb{Z})$  as

$$\frac{F}{2\pi} = n_I f^I + dv = m^I b_I + l_I f^I + dv \quad (3.10)$$

(a cohomology class and its representative we usually denote by the same symbol). This expression needs a more detailed comments:  $v$  is a proper 1-form describing quantum fluctuations. The first two terms of the last right-hand-side corresponding to non-trivial cohomology classes, describe classical (background) solution of the BF-theory,

$$\frac{F_{\text{cl}}}{2\pi} = n_I f^I = m^I b_I + l_I f^I. \quad (3.11)$$

In this formula  $\{b_I\}$  is basis of lattice (group)  $\Lambda = H^2(M, \partial M, \mathbb{Z})$  while  $\{f^I\}$ , basis of the dual lattice (group)  $\Lambda^\# = H^2(M, \mathbb{Z})$ , see subsection 2.3. Since  $\Lambda$  is a sublattice of  $\Lambda^\#$  and since  $i_\partial^*(f^I) = t^I$  are generators of the subgroup  $\mathbb{Z}_{p(I)}$  of the discriminant group  $T(\Lambda) = \Lambda^\#/\Lambda = H^2(M, \mathbb{Z})/H^2(M, \partial M, \mathbb{Z})$ , an arbitrary element of  $\Lambda^\#$  can be represented as a sum of some elements  $m^I b_I$  of the lattice  $\Lambda$  ( $m^I \in \mathbb{Z}$ ) and of such a linear combination  $l_I f^I$ , so that  $l_I \in \overline{0, p(I) - 1}$ . The restriction on the values of the coefficients  $l_I$  is due to  $p(I) f^I \in \Lambda$  (there is no summation in  $I$ ).

Now note that  $i_\partial^*(l_I f^I) = l_I t^I \in \overline{H^2(\partial M, \mathbb{Z})}$ , then due to the isomorphism (2.13) the set of numbers  $l_I \in \overline{0, p(I) - 1}$  determines the bundle  $P_\partial$  on the boundary  $\partial M$  which should be fixed in the calculation of a transition amplitude. Thus in this calculation the summation has to be done over the sets

$m^I \in \mathbb{Z}$ , *i.e.* the summation over  $P \in \text{Princ}(M) \cong H^2(M, \mathbb{Z})$  reduces to that over  $P \in \text{Princ}(M, \partial M) \cong H^2(M, \partial M, \mathbb{Z})$  which coincides with the result in [13].

Passing to the procedure of calculation of the transition amplitude we see that the equation of motion for  $B$  (3.2) is in fact an algebraic constraint which we can substitute back to (3.1) in order to obtain a more usual form of the BF-action [6]

$$S = \frac{1}{4\pi\lambda} \int_M F \wedge F. \quad (3.12)$$

Inserting the expression (3.10) into (3.12), we find

$$S(\lambda, \bar{m}, \underline{l}) = \frac{\pi}{\lambda} (m^I + Q^{IJ} l_J) Q_{IK} (m^K + Q^{KL} l_L) \quad (3.13)$$

where  $\bar{m} = \{m^I | I \in \overline{1, r}\}$ ,  $\underline{l} = \{l_I | I \in \overline{1, r}\}$ , and  $Q_{IJ} = \langle b_I, b_J \rangle_{\mathbb{Z}} = \int_M b_I \wedge b_J$ ,  $Q^{IJ} = \langle f^I, f^J \rangle_{\mathbb{Q}} = \int_M f^I \wedge f^J$  are integer and rational intersection matrices defined in subsection 2.3; remember that  $Q_{IJ} Q^{JK} = \delta_I^K$ . Note that the quantum fluctuations  $v$  give zero contributions in (3.13) due to the Stokes theorem

$$\int_M dv \wedge dv = \int_{\partial M} v \wedge dv = 0 \quad (3.14)$$

and to the normal boundary conditions (3.9). Thus we see that the transition amplitude (partition function corresponding to the graph cobordism  $M$  in Euclidean regime) reads

$$Z(\lambda, \underline{l}) = \frac{1}{c} \sum_{m^I \in \mathbb{Z}} \exp[-S(\lambda, \bar{m}, \underline{l})]$$

where the constant  $c = 1$  due to (3.14), so that

$$Z(\lambda, \underline{l}) = \sum_{m^I \in \mathbb{Z}} \exp \left[ -\frac{\pi}{\lambda} (m^I + Q^{IJ} l_J) Q_{IK} (m^K + Q^{KL} l_L) \right]. \quad (3.15)$$

This is recognizable as a sort of theta function associated with the flux lattice  $\Lambda + Q\underline{l}$  since  $m^I b_I \in \Lambda$ . To clarify this question, we define the analogues of electric and magnetic fluxes related to the field strength  $\frac{F_{\text{el}}}{2\pi} \in H^2(M, \mathbb{Z})$  through a set of homologically non-trivial 2-cycles. First note that in the case of a graph cobordism  $M$  both integer intersection form (2.26) and rational intersection form (2.23) are rigorously determined as

$$\langle \ , \ \rangle_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H^2(M, \partial M, \mathbb{Z}) \rightarrow \mathbb{Z} \quad (3.16)$$

and

$$\langle \ , \ \rangle_{\mathbb{Q}} : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \rightarrow \mathbb{Q}, \quad (3.17)$$

respectively. Due to the Poincaré–Lefschetz duality in the form (2.24) and (2.25), there are two induced pairings

$$\langle \ , \ \rangle_{\mathbb{Z}} : H^2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z} \quad (3.18)$$

and

$$\langle \ , \ \rangle_{\mathbb{Q}} : H^2(M, \mathbb{Z}) \times H_2(M, \partial M, \mathbb{Z}) \rightarrow \mathbb{Q} \quad (3.19)$$

which enable one to determine the fluxes of the field strength  $\frac{F_{\text{cl}}}{2\pi}$  through non-trivial absolute and relative 2-cycles, respectively.

Let us introduce a basis  $\{\phi_I | I \in \overline{1, r}\}$  of homologically non-trivial 2-cycles in  $\Lambda \cong H_2(M, \mathbb{Z})$  dual to the basis  $\{f^I\}$  in  $\Lambda^\# \cong H^2(M, \mathbb{Z})$  with respect to the pairing (3.18) in the sense that

$$\langle f^I, \phi_J \rangle_{\mathbb{Z}} = \delta_J^I. \quad (3.20)$$

Moreover, using once again the Poincaré–Lefschetz duality (2.24), we come from (3.18) to another induced pairing

$$\langle \ , \ \rangle_{\mathbb{Z}}^{\text{hom}} : H_2(M, \partial M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z} \quad (3.21)$$

which gives the usual intersection numbers between the 2-cycles of  $H_2(M, \partial M, \mathbb{Z})$  and  $H_2(M, \mathbb{Z})$  [14]. Now, we can define a basis  $\{\beta^I | I \in \overline{1, r}\}$  of homologically non-trivial relative 2-cycles in  $\Lambda^\# \cong H_2(M, \partial M, \mathbb{Z})$  dual to the basis  $\{\phi_I\}$  with respect to the pairing (3.21) in the sense that

$$\langle \beta^I, \phi_J \rangle_{\mathbb{Z}}^{\text{hom}} = \delta_J^I, \quad (3.22)$$

*i.e.* dual in the sense of Poincaré–Lefschetz. In the definition of analogues of the electric and magnetic fluxes we apply the Poincaré–Lefschetz duality instead of the Hodge duality used in the ordinary electrodynamics with a theta term [7, 8, 10, 11].

We define the “electric” fluxes of field strength as fluxes through the homologically non-trivial 2-cycles  $\phi_I \in \Lambda$  using the scalar product (3.18) as

$$\Phi_I^{(\text{el})}(\bar{m}, \underline{l}) := \left\langle \frac{F_{\text{cl}}}{2\pi}, \phi_I \right\rangle_{\mathbb{Z}} = Q_{IJ} m^J + l_I \in \mathbb{Z}. \quad (3.23)$$

We analogously define “magnetic” fluxes of field strength as fluxes through the homologically non-trivial relative 2-cycles  $\beta^I \in \Lambda^\#$  using the scalar product (3.19) as

$$\Phi_{(\text{mag})}^I(\bar{m}, \underline{l}) := \left\langle \frac{F_{\text{cl}}}{2\pi}, \beta^I \right\rangle_{\mathbb{Q}} = m^I + Q^{IJ} l_J \in \mathbb{Q}. \quad (3.24)$$

The expressions of the fluxes in terms of “quantum numbers”  $m^I$  and  $l_I$  in formulae (3.23) and (3.24) follow from the relation (3.11) and from the duality of corresponding bases. Thus the transition amplitude (3.15) can be rewritten as

$$Z(\lambda, \underline{l}) = \sum_{\bar{m} \in \Lambda} \exp \left[ -\frac{\pi}{\lambda} \Phi_{(\text{mag})}^I(\bar{m}, \underline{l}) Q_{IK} \Phi_{(\text{mag})}^K(\bar{m}, \underline{l}) \right]. \quad (3.25)$$

Consequently, this transition amplitude describes the strong coupling of “magnetic” fluxes (3.24) since the product  $\frac{1}{\lambda} Q_{IJ}$  plays the rôle of coupling constants’ matrix. Here  $Q_{IJ}$  is the integer intersection matrix of the cobordism  $M$  (with all non-zero elements being  $> 1$ ), and the scale factor  $\frac{1}{\lambda} \geq 1$  (to be shown in subsection 3.3).

*Observation 3.1.* Note that if the lattice  $\Lambda$  is self-dual ( $\Lambda \cong \Lambda^\#$ ), the “electric” and “magnetic” fluxes (in the sense of our definition) mutually coincide, thus one has to introduce the Hodge operator, which presumes existence of metric (and we are trying to avoid this), to distinguish between these two types of fluxes. Thus in the consideration of a closed 4-manifold ( $\partial M = 0$ ) our model becomes trivial, and the non-triviality of our approach is due to the substantiveness of the exact cohomological sequence (2.22) which degenerates into isomorphism,

$$0 \rightarrow H^2(M, \partial M, \mathbb{Z}) \xrightarrow{j^*} H^2(M, \mathbb{Z}) \rightarrow 0,$$

if  $\partial M = 0$  or even  $H^2(\partial M, \mathbb{Z}) = 0$ .

To determine the behaviour of  $Z(\lambda, \underline{l})$  under the transformation  $\lambda \rightarrow 1/\lambda$  we use the same trick as Olive and Alvarez [10]. Using the Poisson summation formula in matrix form [37]

$$\begin{aligned} \sum_{\bar{m} \in \Lambda} \exp [-\pi(\bar{m} + \bar{x}) \cdot A \cdot (\bar{m} + \bar{x})] = \\ (\det A)^{-1/2} \sum_{\underline{n} \in \Lambda^\#} \exp [-\pi \underline{n} \cdot A^{-1} \cdot \underline{n} + 2\pi i \underline{n} \cdot \bar{x}], \end{aligned}$$

we can rewrite the transition amplitude (3.15) in terms of the weak coupling between “electric” fluxes (3.23) as

$$\left. \begin{aligned} Z(\lambda, \underline{l}) &= \lambda^{r/2} (\det Q_{IJ})^{-1/2} \times \\ &\sum_{n_I \in \mathbb{Z}} \exp \left[ \pi \lambda \left( 2i n_I Q^{IJ} l_J - n_I Q^{IJ} n_J \right) \right] = \\ &\lambda^{r/2} (\det Q_{IJ})^{-1/2} \sum_{\underline{l}' \in T(\Lambda)} \exp \left( 2\pi i \lambda l'_I Q^{IJ} l_J \right) \times \\ &\sum_{\bar{m} \in \Lambda} \exp \left[ -\pi \lambda \left( Q_{IJ} m^J + l'_I \right) Q^{IK} \left( Q_{KL} m^L + l'_K \right) \right] = \\ &\lambda^{r/2} (\det Q_{IJ})^{-1/2} \sum_{\underline{l}' \in T(\Lambda)} \exp \left( 2\pi i \lambda l'_I Q^{IJ} l_J \right) \times \\ &\sum_{\bar{m} \in \Lambda} \exp \left[ -\pi \lambda \Phi_I^{(\text{el})}(\bar{m}, \underline{l}') Q^{IK} \Phi_K^{(\text{el})}(\bar{m}, \underline{l}') \right]. \end{aligned} \right\} \quad (3.26)$$

In the second and third parts of this formula we used the expressions (3.11) in the form

$$\frac{F_{\text{cl}}}{2\pi} = n_I f^I = m^I b_I + l'_I f^I = (Q_{IJ} m^J + l'_I) f^I = \Phi_I^{(\text{el})}(\bar{m}, \underline{l}') f^I \quad (3.27)$$

and  $b_I = Q_{IJ} f^J$  [14]. The prime in  $l'_I$  (with respect to which the summation is performed) distinguishes it from the fixed  $l_I$ , while the symbolic notation  $\underline{l}' \in T(\Lambda)$  means that the summation runs over all collections  $\{l'_I\}$  which determine the elements  $l'_I t^I$  of the discriminant group  $T(\Lambda)$ , *i.e.*  $l'_I \in 1, p(I) - 1$ .

The transition amplitudes  $Z(\lambda, \underline{l})$  can be considered as  $|T(\Lambda)|$  modifications of generalized theta functions. Comparing the expressions (3.25) and (3.26), one finds the relation

$$Z(\lambda, \underline{l}) = \frac{\lambda^{r/2}}{\sqrt{|T(\Lambda)|}} \sum_{\underline{l}' \in T(\Lambda)} \exp \left( 2\pi i \lambda l'_I Q^{IJ} l_J \right) Z \left( \frac{1}{\lambda}, \underline{l}' \right) \quad (3.28)$$

which can be regarded as action of a variant of the Montonen–Olive duality transformation [38], that is, a  $(\lambda \rightarrow \frac{1}{\lambda})$ -analogue of the  $S$ -transformation  $\tau \rightarrow -\frac{1}{\tau}$  of electric-magnetic duality [7, 8] applied to the transition amplitude in the BF-model (*cf.* section 6 in [10]). Thus the transition amplitudes written as (3.25) and (3.26), are expressed by means of  $|T(\Lambda)|$  “theta functions” depending on the principal  $U(1)$ -bundles  $P_{\partial}$  (on the boundary  $\partial M$ )



classified by Chern classes  $c_\partial = l_I t^I \in T(\Lambda)$  where  $\{t^I\}$  are generators of  $H^2(\partial M, \mathbb{Z})$ . Since for our graph cobordisms  $T(\Lambda) = H^2(\partial M, \mathbb{Z}) = \oplus_{I=1}^r \mathbb{Z}_{p(I)}$  (pure torsion), all principal  $U(1)$ -bundles on  $\partial M$  are flat [12]:

$$\text{Princ}_0(\partial M) \stackrel{c_\partial}{\cong} H^2(\partial M, \mathbb{Z}) = \bigoplus_{I=1}^r \mathbb{Z}_{p(I)}. \quad (3.29)$$

Due to (2.16) for our case, all flat principal  $U(1)$ -bundles on  $\partial M$  (fixed by the sets  $\{l_I \in \overline{0, p(I) - 1} | I \in \overline{1, r}\}$  also known as *rotation numbers* [32, 25]) are extendable to  $M$ . These extensions are parametrized by the group of relative bundles  $\text{Princ}(M, \partial M) \stackrel{c_{\text{rel}}}{\cong} H^2(M, \partial M, \mathbb{Z})$  (or equivalently by the set of integer parts  $\{m^I \in \mathbb{Z} | I \in \overline{1, r}\}$  of rational “magnetic” fluxes  $\Phi_{(\text{mag})}^I$ ) in the one-to-one manner due to (2.17).

*Observation 3.2.* The partition sums (3.25) and (3.26) converge if the intersection matrices  $Q_{IJ}$  and  $Q^{IJ}$  are positive definite. In section 4 examples of graph cobordisms satisfying this condition will be given.

*Observation 3.3.* The passage from (3.25) to (3.26) for the transition amplitudes corresponds to an interchange of strong and weak couplings: (3.25) involves the coupling constants matrix  $\frac{1}{\lambda} Q_{IJ}$  whose non-zero elements are  $> 1$  (strong coupling); this formula is related to interaction of “magnetic” fluxes passing through homologically non-trivial closed 2-surfaces  $\beta^I$ , while the expression (3.26) contains the coupling constants matrix  $\lambda Q^{IJ}$  whose non-zero elements are  $< 1$  (weak coupling), and it is related to interaction of “electric” fluxes  $\Phi_I^{(\text{el})}(\bar{m}, \underline{l})$  (through homologically non-trivial closed 2-cycles  $\phi_I$ ) which mimic presence of quantized “electric charges” always being integer according to (3.23). In the same manner, the “magnetic” fluxes captured by homologically non-trivial 2-cycles  $\beta^I$  can be interpreted as effective quantized “magnetic charges” possessing rational values since  $\Phi_{(\text{mag})}^I(\bar{m}, \underline{l}) \in \mathbb{Q}$ . Note that these effective “magnetic charges” pertain to a specific fixed subset of rational numbers with a finite collection of different denominators composed only by products of the topological invariants  $p'_1, \dots, p'_{2r+1}$  of the graph cobordism  $M$  (see subsection 2.3). Thus from the BF-analogue of the generalized Dirac quantization conditions (3.7) and (3.8) it follows that the “electric charges” come to be integer, while the “magnetic charges” are found to be rational.

*Observation 3.4.* It is interesting to note that the same formulae (3.23) and (3.24) can be obtained from fluxes of basic 2-cocycles  $b_I \in H^2(M, \partial M, \mathbb{Z}) = \Lambda$

and  $f^I \in H^2(M, \mathbb{Z}) = \Lambda^\#$  through a general 2-cycle  $Z_2$  which is possible to represent in the form

$$Z_2 = m^J \phi_J + l_J \beta^J, \quad (3.30)$$

using the same ideas as for writing the expression (3.11). Then the flux of the basic field  $b_I$  through  $Z_2$  reads

$$\langle b_I, m^J \phi_J + l_J \beta^J \rangle = m^J Q_{IJ} + l_J \delta_I^J = \Phi_I^{(\text{el})}(\bar{m}, \underline{l}) \in \mathbb{Z} \quad (3.31)$$

which coincides with the “electric” flux (3.23). Thus the matrix element  $Q_{IJ} = \langle b_I, \phi_J \rangle \in \mathbb{Z}$  may be interpreted as an elementary “electric charge” imitated by the flux of basic “electric field”  $b_I$  through the basic 2-cycle  $\phi_J \in \Lambda$ . Analogously,  $\delta_I^J = \langle b_I, \beta^J \rangle$  can be understood as an elementary “electric charge” simulated by the flux of basic 2-cocycle  $b_I$  through the basic 2-cycle  $\beta^J \in \Lambda^\#$ . In the same way the flux of the dual basic field  $f^I \in \Lambda^\#$  through  $Z_2$  is

$$\langle f^I, m^J \phi_J + l_J \beta^J \rangle = m^J \delta_J^I + l_J Q^{IJ} = \Phi_{(\text{mag})}^I(\bar{m}, \underline{l}) \in \mathbb{Q}, \quad (3.32)$$

being the same as the “magnetic” flux (3.24). This leads to the interpretation of matrix element  $Q^{IJ} = \langle f^I, \beta^J \rangle \in \mathbb{Q}$  as an elementary “magnetic charge” imitated by the flux of basic “magnetic field”  $f^I$  captured by the basic 2-cycle  $\beta^J \in \Lambda^\#$ , and of matrix element  $\delta_J^I = \langle f^I, \phi_J \rangle$  as an elementary “magnetic charge” imitated by the flux of basic 2-cocycle  $f^I$  through the basic 2-cycle  $\phi_J \in \Lambda$ .

This picture resembles well aged ideas of Wheeler and Misner [39, 40] about “charges without charges” when the field strength lines are captured by topological handles (wormholes = topological non-trivialities of the spacetime manifold). It occurs that the spacetime topology has to be unexpectedly complex when one is trying to reproduce certain characteristic features of the real universe. In section 4 we propose a concrete model exemplifying the possibility of dealing with the problems of the number of fundamental interactions in the universe as well as the hierarchy of their coupling constants on the purely topological level, but using rather complicated four-manifolds (graph cobordisms).

### 3.3 Upper bounds of the scale factor

In our model, the generalized Dirac quantization conditions (3.7) and (3.8), together with the exactness of cohomological sequence (2.22), give upper

bounds of the scale factor  $\lambda$  introduced in the action (3.1). These bounds are determined in terms of the topological invariants of the graph cobordism  $M$ , but they also depend on the Chern classes  $c_\partial = l_I t^I$  of the principal  $U(1)$ -bundles  $P_\partial$  which are fixed on the boundary  $\partial M$ . Note that the parameter  $\lambda$  does determine the scale factor of the coupling constants matrix as  $\lambda Q^{IJ}$  for weak coupling in (3.26) and  $\frac{1}{\lambda} Q_{IJ}$  for strong coupling in (3.25) where the rational and integer intersection matrices  $Q^{IJ}$  and  $Q_{IJ}$  give the hierarchy of the corresponding coupling constants. Due to exactness of the cohomological sequence (2.22) and since  $H^2(\partial M, \mathbb{Z})$  is a pure torsion for any class  $\frac{F_{\text{cl}}}{2\pi} \in H^2(M, \mathbb{Z})$  (3.11), there exists such a minimal positive integer  $q_0$  that  $q_0 \frac{F_{\text{cl}}}{2\pi}$  is a certain element  $\frac{B_{\text{cl}}}{2\pi}$  of the group  $H^2(M, \partial M, \mathbb{Z})$  [32, 14], *i.e.*

$$q_0 F_{\text{cl}}/2\pi = B_{\text{cl}}. \quad (3.33)$$

(It is obvious that for any positive integer  $k$  the class  $kq_0 F_{\text{cl}}/2\pi$  will certainly belong to the group  $H^2(M, \partial M, \mathbb{Z})$ .) Comparing the relation (3.33) and the classical constraint equation (3.2),  $F = \lambda B$ , we find the upper bound of  $\lambda$ , namely  $\lambda_0 = 1/q_0$ . Moreover, the scale factor  $\lambda$  is quantized in the sense that  $\lambda_{k-1} = 1/(kq_0)$ ,  $k \in \mathbb{N}$ . The value of  $q_0$  can be found from the expansion of solutions of the equation (3.33) with respect to the bases  $\{b_I\}$  and  $\{f^I\}$  of the groups  $H^2(M, \partial M, \mathbb{Z})$  and  $H^2(M, \mathbb{Z})$ , respectively,

$$\frac{B_{\text{cl}}}{2\pi} = k^I b_I, \quad k^I \in \mathbb{Z}, \quad (3.34)$$

$$\frac{F_{\text{cl}}}{2\pi} = m^I b_I + l_I f^I, \quad m^I \in \mathbb{Z}, l_I \in \overline{0, p(I) - 1} \quad (3.35)$$

(generalized Dirac quantization conditions). A substitution of these solutions into (3.33) yields

$$k^I b_I = q_0 (m^I b_I + l_I f^I). \quad (3.36)$$

This condition would be satisfied if the term  $q_0 l_I f^I$  pertained to the group  $H^2(M, \partial M, \mathbb{Z})$ , *i.e.* if such a collection  $\{s^I | I \in \overline{1, r}\}$  of integers could be found that

$$q_0 l_I f^I = s^I b_I. \quad (3.37)$$

A pairing of the last equation with the basis elements  $f^J$  of the group  $H^2(M, \mathbb{Z})$  yields

$$s^I = q_0 Q^{IJ} l_J. \quad (3.38)$$

Here the problem consists of rationality of  $Q^{IJ}$  while  $\{s^I\}$  is a collection of integers. Note that the cobordisms under consideration correspond to graphs shown in figure 8 thus having the only non-zero elements  $Q^{II}$  and  $Q^{I\pm 1}$ . If for some value of  $J$  the rotation number  $l_J = 0$ , the matrix elements  $Q^{JJ}$  and  $Q^{JJ\pm 1}$  do not enter (3.38). If  $l_J \neq 0$ , such elements give non-zero contribution. These elements have the common denominator  $\tilde{P}_J = \text{LCM}(p'_{2J-1}, p'_{2J}, p'_{2J+1})$  where LCM means Least Common Multiple, while  $p'_s$  are the positive integers characterizing the decorated graph  $\Gamma_D$  in figure 8 (see subsections 2.1, 2.3). Thus all terms in the right-hand side of (3.38) will be integers, if

$$q_0 \equiv q_0(\underline{l}) = \text{LCM} \left( \frac{\tilde{P}_J}{\text{GCD}(\tilde{P}_J, l_J)}, \text{ over all } J \text{ such that } l_J \neq 0 \right) \quad (3.39)$$

where GCD means Greatest Common Divisor, and the notation  $q_0(\underline{l})$  takes into account dependence on the rotation numbers  $l_J$ . It is worth being emphasized that the upper bound  $\lambda_0(\underline{l}) = 1/q_0(\underline{l})$  of the scale factor in our BF-model depends not only on topological invariants  $p'_s$  of the graph cobordism  $M$ , but also on the Chern class  $c_\partial = l_I t^I$  which fixes the principal bundle  $P_\partial$  on the boundary  $\partial M$ . Note that  $\lambda_0(\underline{l}) = 1/q_0(\underline{l})$  itself is the upper bound of the sequence of admissible scale factors  $\lambda_{k-1}(\underline{l}) = 1/(kq_0(\underline{l}))$ ,  $k \in \mathbb{N}$ .

*Observation 3.5* From (3.39) one sees that the quantity  $q_0(\underline{l})$  takes its maximum value when all  $l_J \neq 0$  and  $\text{GCD}(\tilde{P}_J, l_J) = 1$ . Then

$$\bar{q}_0 := \max_{\underline{l}} q_0(\underline{l}) = \text{LCM}(\tilde{P}_J, J = \overline{1, r}) = \text{LCM}(p'_s, s = \overline{1, 2r+1}). \quad (3.40)$$

The quantity  $q_0(\underline{l})$  takes its minimum value when all  $l_J = 0$ . It is obvious that in this case  $\underline{q}_0 := \min_{\underline{l}} q_0(\underline{l}) = 1$ . Thus the upper bounds  $\lambda_0(\underline{l})$  of the scale factor in our BF-model take discrete values in the interval

$$\frac{1}{\bar{q}_0} \leq \lambda_0(\underline{l}) \leq 1. \quad (3.41)$$

In section 4 we shall build a cosmological model in which the present-stage universe is characterized by an integer  $\bar{q}_0$  having the order of magnitude  $3.28 \cdot 10^{177}$ .

## 4 The family of graph cobordisms as a sequence of cosmological models

In this section we construct a collection of graph cobordisms interpretable as a sequence of topological changes finally resulting in the state of universe which we identify as its contemporary stage by the number of fundamental interactions and the hierarchy of their coupling constants. We proposed a similar type of model in recent papers [1, 41]. The construction we realize now differs by an additional condition on the four-dimensional topological space playing the rôle of the spacetime manifold: its intersection matrix is demanded to be positive defined (see observation 2.2). This guarantees convergence of partition sums (3.25) and (3.26) in the topological gauge theories built on the graph cobordisms (see section 3).

### 4.1 The basic family of Seifert fibred homology spheres

The basic structure elements of graph cobordisms used in this paper are simple graph four-manifolds with Seifert fibred Brieskorn homology (Bh-) spheres  $\Sigma(a_1, a_2, a_3)$  as boundaries (see subsection 2.1). (Compact locally homogeneous universes with spatial sections homeomorphic to Seifert fibrations were considered at length in [42, 43, 44].) We use only a specific bi-parametric family of Bh-spheres which is defined as follows: First, we introduce the *primary sequence* of Bh-spheres (see [1] for more details). Let  $p_i$  be the  $i$ th prime number in the set of positive integers  $\mathbb{N}$ , *e.g.*  $p_1 = 2$ ,  $p_2 = 3, \dots, p_9 = 23, \dots$ . Then the primary sequence is defined as

$$\{\Sigma(p_{2n}, p_{2n+1}, q_{2n-1}) | n \in \mathbb{Z}^+\} \quad (4.1)$$

where  $q_i := p_1 \cdots p_i$ ,  $\mathbb{Z}^+$  is the set of non-negative integers. The first terms in this sequence with  $n > 0$  (which we really use) are  $\Sigma(2, 3, 5)$  (the Poincaré homology sphere),  $\Sigma(7, 11, 30)$ ,  $\Sigma(13, 17, 2310)$ , and  $\Sigma(19, 23, 510510)$ . We also include in this sequence as its first term ( $n = 0$ ) the usual three-dimensional sphere  $S^3$  (Sf-sphere) with Seifert fibration determined by the mapping  $h_{pq} : S^3 \rightarrow S^2$ , in its turn defined as  $h_{pq}(z_1, z_2) = z_1^p/z_2^q$  [45]. Recall that  $S^3 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1\}$  and  $z_1^p/z_2^q \in \mathbb{C} \cup \{\infty\} \cong S^2$ . In this paper we consider the case  $p = 1$ ,  $q = 2$  and denote this Sf-sphere as  $\Sigma(1, 2, 1)$ , *i.e.*  $p_0 = q_{-1} = 1$  in (4.1). In this notation we use two additional units which correspond to two arbitrary regular fibers. This will enable us to operate

with  $\Sigma(1, 2, 1)$  in the same manner as with other members of the sequence (4.1).

Second, we define  $k^\pm$ -operations for each of Bh-spheres  $\Sigma(a_1, a_2, a_3)$ . To start with, we renumber Seifert's invariants so that  $a_1 < a_2 < a_3$ ; this is always possible since  $a_1, a_2$  and  $a_3$  are pairwise coprime (in the case of the Sf-sphere, we take the order  $\Sigma(1, 1, 2)$ ). The result of  $k^\pm$ -operation acting on  $\Sigma(a_1, a_2, a_3)$  is another Bh-sphere

$$\Sigma_{k_1}^\pm(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}) = \Sigma(a_1, a_2 a_3, k_1 a \pm 1), \quad (4.2)$$

*i.e.* it is the Bh-sphere with Seifert invariants

$$a_1^{(1)} = a_1, a_2^{(1)} = a_2 a_3, a_3^{(1)} = k_1 a \pm 1 \quad (4.3)$$

where  $a = a_1 a_2 a_3, k_1 \in \mathbb{N}$ . The upper index in the parentheses means a single application of the  $k^\pm$ -operation. A repeated application of this operation yields still another Bh-sphere

$$\Sigma_{k_1 k_2}^\pm(a_1^{(2)}, a_2^{(2)}, a_3^{(2)}) = \Sigma(a_1, a_2 a_3 (k_1 a \pm 1), k_2 a (k_1 a \pm 1) \pm 1) \quad (4.4)$$

where  $k_2 \in \mathbb{N}$ ; in general,  $k_2 \neq k_1$ . The  $l$ -fold application of the  $k^\pm$ -operation again gives an Bh-sphere,  $\Sigma_{k_1 \dots k_l}^\pm(a_1^{(l)}, a_2^{(l)}, a_3^{(l)})$  whose invariants are found by induction from the invariants  $a_1^{(l-1)}, a_2^{(l-1)}, a_3^{(l-1)}$ , with arbitrary  $k_l \in \mathbb{N}$ . Note that the least Seifert invariant does not change under  $k^\pm$ -operations ( $a_1^{(l)} = a_1$  for any  $l = 1, 2, \dots$ ) while the two other Seifert invariants depend both on the order (multiplicity) of the  $k^\pm$ -operation fulfilment and on which ( $k^+$  or  $k^-$ )-operation is applied. A hint of such an operation can be found in Saveliev's paper [46].

In [1] we defined only the  $k^+$ -operation in the special case  $k_1 = k_2 = \dots = k_l = 1$  and named it (not quite aptly) "derivative of Bh-sphere". In our new terminology this is the  $1^+$ -operation; its  $l$ -fold application gives the Bh-sphere denoted in [1] as  $\Sigma(a_1^{(l)}, a_2^{(l)}, a_3^{(l)})$ . In the same paper we showed that the application of this operation to the primary sequence (4.1) yields a bi-parametric family of Bh-spheres whose Euler numbers reproduce fairly well the experimental hierarchy of dimensionless low-energy coupling (DLEC) constants of the fundamental interactions in the real universe. For the reader's convenience we concisely reiterate here some results obtained in [1].

This bi-parametric family of Bh-spheres is

$$\left\{ \Sigma(a_{1n}^{(l)}, a_{2n}^{(l)}, a_{3n}^{(l)}) = \Sigma(p_{2n}^{(l)}, p_{2n+1}^{(l)}, q_{2n-1}^{(l)}) | n, l \in \mathbb{Z}^+ \right\}. \quad (4.5)$$

Table 1: Euler number of  $(n, t)$ -family of Sf- and Bh-spheres.

$\begin{smallmatrix} \diagdown & t \\ n & \diagup \end{smallmatrix}$	-4	-3	-2	-1	0	1	2	3	4
0					$5.0 \times 10^{-1}$	$1.7 \times 10^{-1}$	$2.3 \times 10^{-2}$	$5.5 \times 10^{-4}$	$3.1 \times 10^{-7}$
1				$3.3 \times 10^{-2}$	$1.1 \times 10^{-3}$	$1.2 \times 10^{-6}$	$1.3 \times 10^{-12}$	$1.8 \times 10^{-24}$	
2			$4.3 \times 10^{-4}$	$1.9 \times 10^{-7}$	$3.5 \times 10^{-14}$	$1.2 \times 10^{-27}$	$1.5 \times 10^{-54}$		
3		$2.0 \times 10^{-6}$	$3.8 \times 10^{-12}$	$1.5 \times 10^{-23}$	$2.2 \times 10^{-46}$	$4.7 \times 10^{-92}$			
4	$4.5 \times 10^{-9}$	$2.0 \times 10^{-17}$	$4.0 \times 10^{-34}$	$1.6 \times 10^{-67}$	$2.7 \times 10^{-134}$				

(Note that the  $k^\pm$ -operation involves a renumbering of Seifert's invariants such that the inequalities  $a_{1n}^{(l)} < a_{2n}^{(l)} < a_{3n}^{(l)}$  become valid. Thus the collections of Seifert's invariants  $\{a_{1n}^{(l)}, a_{2n}^{(l)}, a_{3n}^{(l)}\}$  and  $\{p_{2n}^{(l)}, p_{2n+1}^{(l)}, q_{2n-1}^{(l)}\}$  are equivalent up to ordering.) In [1] it was shown that to reproduce the hierarchy of the DLEC constants of the known five fundamental interactions (including the cosmological one) it is sufficient to restrict values of parameters as  $n, l \in \overline{0, 4}$ . With this restriction, the Euler numbers of the Bh-spheres family are given in table 1 (the revised table 3 of [1]).

To make the comparison with the experimental hierarchy of DLEC constants (see table 2) easier, we introduced instead of  $l$  a new parameter  $t := l - n$  which plays the rôle of “discrete cosmological time”.

Just at  $t = 0$  ( $l = n$ ) the experimental hierarchy of DLEC constants is reproduced properly. This enables us to consider the ensemble of Bh-spheres (4.5) at  $l = n$

$$E_0 = \left\{ \Sigma(a_{1n}^{(n)}, a_{2n}^{(n)}, a_{3n}^{(n)}) = \Sigma(p_{2n}^{(n)}, p_{2n+1}^{(n)}, q_{2n-1}^{(n)}) | n \in \overline{0, 4} \right\} \quad (4.6)$$

as the basis elements used in constructing the spatial section  $\Sigma_0$  of the contemporary universe by means of the splicing operation. The key factor in this (at first glance, exotic) hypothesis is the fact that the diagonal elements (and eigenvalues) of the rational intersection matrix for the corresponding graph cobordism  $M_0$  (that is, its  $\partial M_0 = \Sigma_0 \sqcup_{s=1}^N (-L(|p'_s|, q'_s))$ ) show the same hierarchy as the Euler numbers, thus reproducing the DLEC constants' hierarchy (see two last columns in table 2). Then it is natural to suppose that at  $t \in \overline{-4, -1}$  the ensembles

$$E_t = \left\{ \Sigma(a_{1n}^{(n+t)}, a_{2n}^{(n+t)}, a_{3n}^{(n+t)}) = \Sigma(p_{2n}^{(n+t)}, p_{2n+1}^{(n+t)}, q_{2n-1}^{(n+t)}) | n \in \overline{-t, 4} \right\} \quad (4.7)$$

of Bh-spheres forming basic elements for gluing (by splicing) spatial sections of the universe on earlier stages characterized, in particular, by a diminishing of the number of fundamental interactions from five at  $t = 0$  to one at  $t = -4$ .

Table 2: Euler numbers *vs.* experimental DLEC constants *vs.* diagonal elements of intersection matrix  $Q^+(0)$  (see subsection 4.3).

$n$	$e\left(\Sigma_n^{(n)}\right)$	Interaction	$\alpha_{\text{exper}}$	$Q^{+II}(0)$	$I$
0	0.5	strong	1	$9.69 \times 10^{-1}$	1
1	$1.07 \times 10^{-3}$	electromagnetic	$7.20 \times 10^{-3}$	$7.21 \times 10^{-3}$	2
2	$3.51 \times 10^{-14}$	weak	$3.04 \times 10^{-12}$	$1.76 \times 10^{-12}$	3
3	$2.17 \times 10^{-46}$	gravitational	$2.73 \times 10^{-46}$	$3.68 \times 10^{-44}$	4
4	$2.70 \times 10^{-134}$	cosmological	$< 10^{-120}$	$2.66 \times 10^{-134}$	5

**Notes:** **1.** The dimensionless strong interaction constant is  $\alpha_{\text{st}} = G/\hbar c$ ,  $G$  characterizes the strength of the coupling of the meson field to the nucleon. **2.** The fine structure (electromagnetic) constant is  $\alpha_{\text{em}} = e^2/\hbar c$ . **3.** The dimensionless weak interaction constant is  $\alpha_{\text{weak}} = (G_F/\hbar c)(m_e c/\hbar)^2$ ,  $G_F$  being the Fermi constant ( $m_e$  is mass of electron). **4.** The dimensionless gravitational coupling constant is  $\alpha_{\text{gr}} = G_N m_e^2/\hbar c$ ,  $G_N$  being the Newtonian gravitational constant. **5.** The cosmological constant  $\Lambda$  multiplied by the squared Planckian length is  $\alpha_{\text{cosm}} = \Lambda G_N \hbar/c^3$ . The mentioned dimensionless constants (except the cosmological one) are also known as Dyson numbers.

*Observation 4.1.* In this paper we shall not consider splice diagrams occurring for  $t > 0$ . Just note that in accordance with table 1 the number of interactions should diminish from five to one with the increase of the parameter  $t$  from 0 to 4, but the intersection matrices (related to the coupling constants ones) are different in ascending and descending stages.

*Observation 4.2.* It is worth being observed that in our scheme the five (low energy) interactions are related to the first nine prime numbers as (1,2), (3,5), (7,11), (13,17), (19,23). To obtain any new interaction, one has to attach a new pair of prime numbers to the preceding set. For example, taking the next pair (29,31), we come with the same algorithm to a new coupling constant of the order of magnitude  $\alpha_6 \approx 10^{-361}$ . Thus our model answers the question we did not even put: How many fundamental interactions may really exist in the universe? Our model predicts an infinite number of interactions due to the infinite succession of prime numbers. We simply cannot detect too weak interactions beginning with  $\alpha_6$  since all subsequent are even weaker:  $\alpha_7 \approx 10^{-916}$ , *etc.* [47].



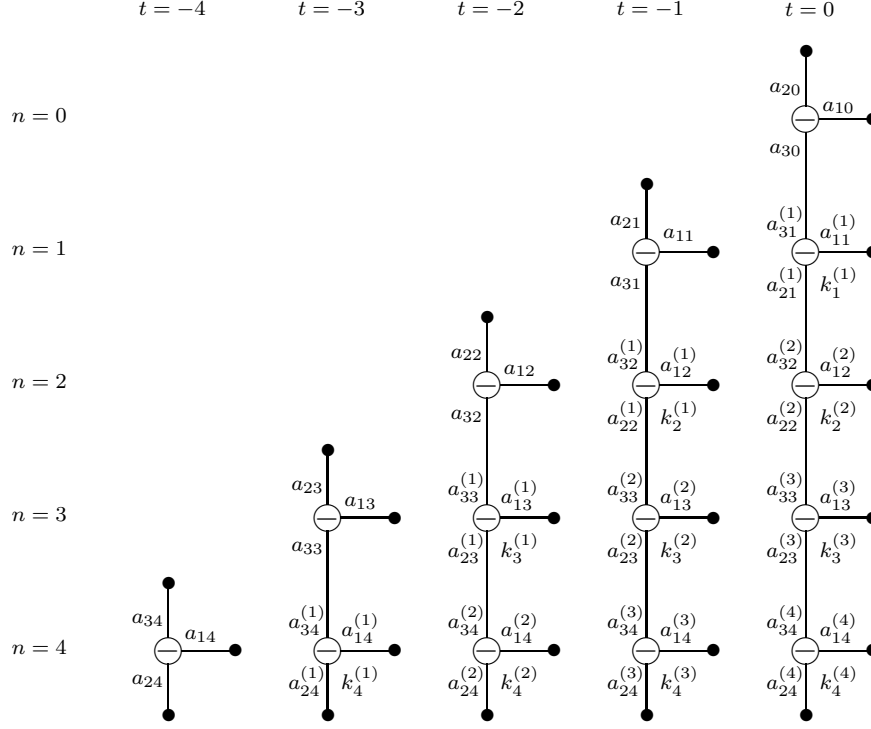


Figure 9: The splice diagram of different states of the universe.

## 4.2 The construction of graph cobordisms

The splice diagrams corresponding to states of the universe at the cosmological time  $t \in \overline{-4, 0}$  are shown in figure 9 where we consider them as subdiagrams being parts of a disjoint total splice diagram. To any of these subdiagrams one associates (in accordance with the well-known algorithm, [26, 15, 17]) the graph cobordism  $M_D$  (constructed in Observation 2.2, subsection 2.1) whose intersection form is positive definite iff for each edge joining two nodes the edge determinant is positive. The *edge determinant*  $\det(e_{mn})$  of an edge joining two nodes is the product of the two weights on the edge minus the product of the weights adjacent to the edge [17]. In our case this criterion means that in any portion of splice diagram shown in figure 10, *i.e.*

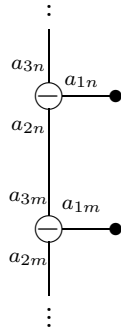


Figure 10: A portion of a splice diagram.

for any edge  $e_{mn}$  one has

$$\det(e_{mn}) = a_{2n}a_{3m} - a_{1n}a_{3n}a_{1m}a_{2m} > 0, \quad (4.8)$$

*cf.* subsection 2.1 where a more general form of the edge determinant is given,  $\det(e_{mn}) = -p$ ,  $p$  being defined in (2.3). Taking certain collections of integers  $k_1, \dots, k_4$  (which participate in  $k^\pm$ -operations), one gets positive definite intersection forms for all splice diagrams shown in figure 9. Note that when  $k_1 = k_2 = k_3 = k_4 = 1$ , all intersection matrices have indefinite signature, thus convergence of transition amplitudes (3.25) and (3.26) for corresponding graph cobordisms is not ensured. Below we confine ourselves to the  $k^+$ -operation (for the  $k^-$ -operation we shall only give the final result in the Appendix).

The total disconnected diagram in figure 9 consists of five connected splice subdiagrams. Naturally, there exists an ambiguity in the splice operation (related to this type of diagrams). The diagram in figure 9 contains fifteen nodes, each of them having three adjacent edges. Thus one can glue  $3^{15}$  different graph cobordisms. Moreover, there is an infinite set of integers  $k_i$  which guarantee positive definiteness of the intersection matrices of respective cobordisms. It is however possible to fix a unique gluing procedure imposing a minimality condition on the coefficients  $k_i$  at each level of realization of the  $k^+$ -operation. In particular, this condition immediately yields a conclusion that the vertices (leaves) corresponding to minimal Seifert invariants ( $a_1^{(l)}$ ) remain free (not subjected to slicing). Applying the  $k^+$ -operation to all Bh-

spheres of the primary sequence

$$\{\Sigma(a_{1n}, a_{2n}, a_{3n}) | n \in \overline{1, 4}\}, \quad (4.9)$$

we find the minimal  $k_n^{(1)}$  for which the conditions  $a_{3n}a_{3,n+1}^{(1)} - a_{1n}a_{2n}a_{1,n+1}^{(1)}a_{2,n+1}^{(1)} > 0$ ,  $n \in \overline{0, 3}$  are satisfied. Thus we unambiguously fixed the collection of the first-level Bh-spheres, *i.e.* those with the parameter  $l = 1$ :

$$\{\Sigma_{k_n^{(1)}}(a_{1n}^{(1)}, a_{2n}^{(1)}, a_{3n}^{(1)}) | n \in \overline{1, 4}\}. \quad (4.10)$$

Now we execute the first splicing procedure (along the upper vertical edges between nodes in figure 9):

$$\Sigma(a_{1n}, a_{2n}, a_{3n}) \frac{S}{S'} \Sigma_{k_{n+1}^{(1)}}(a_{1,n+1}^{(1)}, a_{2,n+1}^{(1)}, a_{3,n+1}^{(1)}) \quad (4.11)$$

where  $n \in \overline{0, 3}$ ,  $S = S_{a_{3n}}$ ,  $S' = S_{a_{3,n+1}}$ . The same algorithm is applied to determine the collection of the second-level Bh-spheres, and further by induction the  $l$ th-level Bh-spheres:

$$\{\Sigma_{k_n^{(1)} \dots k_n^{(l)}}(a_{1n}^{(l)}, a_{2n}^{(l)}, a_{3n}^{(l)}) | n \in \overline{l, 4}\}. \quad (4.12)$$

In each step, there is executed the splice operation according to the diagram in figure 9.

Consequently, we obtain the five connected subdiagrams  $\Delta^+(t)$  where superscript  $+$  corresponds to the use of  $k^+$ -operation with minimization of parameters  $k_n^{(l)}$  at each step. According to the procedure described in the subsection 2.1, the corresponding decorated plumbed graphs  $\Gamma_D^+(t)$  are constructed. These decorated graphs codify the definite graph cobordisms  $M_D^+(t)$ ,  $t \in \overline{-4, 0}$ , which are interpreted as the spacetime manifolds corresponding to different values of cosmological time parameter  $t$ . The boundaries of these cobordisms are represented as follows:

$$\partial M_D^+(t) = \left( - \bigsqcup_{s=1}^{N(t)} L(|p'_s(t)|, q'_s(t)) \right) \bigsqcup \Sigma^+(t). \quad (4.13)$$

(see the expression (2.8)). It is worth being underlined that both  $\mathbb{Z}$ -homology sphere  $\Sigma^+(t)$  and the collection of lens spaces  $L(|p'_s(t)|, q'_s(t))$  depend on the cosmological time  $t$ . Note that among the lens spaces forming the boundaries

of different cobordisms  $M_D^+(t)$  there exist mutually homeomorphic, namely  $L(a_1^{(l)}, b_1^{(l)})$ , since  $a_1^{(l)} = a_1$  for  $\forall l \in \mathbb{N}$ . By means of successive pairwise gluing together these lens spaces, it is possible to form a cobordism  $M_{\text{total}}$  which connects the initial state of universe, with a spatial section  $\Sigma^+(-4)$ , to the final one with a spatial section  $\Sigma^+(0)$ . The cobordism  $M_{\text{total}}$  will include all intermediate stages with the following sequence of spatial sections

$$\Sigma^+(-4) \rightarrow \Sigma^+(-3) \rightarrow \Sigma^+(-2) \rightarrow \Sigma^+(-1) \rightarrow \Sigma^+(0). \quad (4.14)$$

This is accompanied by creation and annihilation of a certain set of disjoint lens spaces of the type  $L(|p'_s(t)|, q'_s(t))$  which have no homeomorphic counterparts. This procedure is outlined in [1].

### 4.3 Discussion: coupling constants of fundamental interactions as cosmological circumstances

In order to deepen our physical discussion, we give below the calculations results for rational intersection matrices  $Q^{+IJ}(t)$  ( $I, J = \overline{1, 5+t}$ ), their eigenvalues, and determinants, corresponding to graph cobordisms  $M_D^+(t)$ :

$$Q^{+IJ}(0) = \begin{pmatrix} \mathbf{9.7 \times 10^{-1}} & 3.1 \times 10^{-2} & 0 & 0 & 0 \\ 3.1 \times 10^{-2} & \mathbf{7.2 \times 10^{-3}} & 1.4 \times 10^{-8} & 0 & 0 \\ 0 & 1.4 \times 10^{-8} & \mathbf{1.8 \times 10^{-12}} & 1.9 \times 10^{-29} & 0 \\ 0 & 0 & 1.9 \times 10^{-29} & \mathbf{3.7 \times 10^{-44}} & 3.1 \times 10^{-89} \\ 0 & 0 & 0 & 3.1 \times 10^{-89} & \mathbf{2.7 \times 10^{-134}} \end{pmatrix},$$

$$\lambda_I^+(0) = \{9.7 \times 10^{-1}, 6.2 \times 10^{-3}, 1.7 \times 10^{-12}, 3.7 \times 10^{-44}, 6.4 \times 10^{-139}\},$$

$$\det Q^{+IJ}(0) = 2.4 \times 10^{-196};$$

$$Q^{+IJ}(-1) = \begin{pmatrix} 8.3 \times 10^{-2} & 1.1 \times 10^{-4} & 0 & 0 \\ 1.1 \times 10^{-4} & 9.6 \times 10^{-6} & 1.2 \times 10^{-14} & 0 \\ 0 & 1.2 \times 10^{-14} & 2.5 \times 10^{-21} & 2.0 \times 10^{-44} \\ 0 & 0 & 2.0 \times 10^{-44} & 1.6 \times 10^{-67} \end{pmatrix},$$

$$\lambda_I^+(-1) = \{8.3 \times 10^{-2}, 9.5 \times 10^{-6}, 2.5 \times 10^{-21}, 9.7 \times 10^{-72}\},$$

$$\det Q^{+IJ}(-1) = 1.9 \times 10^{-98};$$

$$\begin{aligned}
Q^{+IJ}(-2) &= \begin{pmatrix} 3.0 \times 10^{-3} & 1.5 \times 10^{-7} & 0 \\ 1.5 \times 10^{-7} & 6.6 \times 10^{-10} & 5.1 \times 10^{-22} \\ 0 & 5.1 \times 10^{-22} & 4.0 \times 10^{-34} \end{pmatrix}, \\
\lambda_I^+(-2) &= \{3.0 \times 10^{-3}, 6.5 \times 10^{-10}, 7.9 \times 10^{-37}\}, \\
\det Q^{+IJ}(-2) &= 1.5 \times 10^{-48}; \\
Q^{+IJ}(-3) &= \begin{pmatrix} 2.2 \times 10^{-6} & 2.1 \times 10^{-12} \\ 2.1 \times 10^{-12} & 2.2 \times 10^{-17} \end{pmatrix}, \\
\lambda_I^+(-3) &= \{2.2 \times 10^{-6}, 2.0 \times 10^{-17}\}, \\
\det Q^{+IJ}(-3) &= 4.4 \times 10^{-23}; \\
Q^{+IJ}(-4) &= (4.5 \times 10^{-9}).
\end{aligned}$$

(In Appendix we shall give the rational intersection matrices  $Q^{-IJ}(t)$  corresponding to the cobordisms  $M_D^-(t)$  obtained from the splice diagrams in figure 9 by an application of the  $k^-$ -operation while minimizing the parameter  $k^-$  at each step.)

Note that all elements  $Q^{+IJ}(t)$  in these matrices are rational; they are given here up to two significant digits. The inverse matrices  $Q_{IJ}^+(t)$  are integer. The inversion of the rational intersection matrices with the help of the MAPLE program is an excellent test of the correctness of their calculation according to algorithms described in [15, 26], since any error leads to non-integer elements in resulting matrices  $Q_{IJ}^+(t)$ . Recalling the interpretation of rational intersection matrices  $\lambda Q^{+IJ}(t)$  as the coupling constants of “electric” fluxes proposed in subsection 3.2 (see *Observation 3.3*) we observe that the diagonal elements of  $5 \times 5$  matrix  $Q^{+IJ}(0)$  (see boldface numbers) reproduce rather exactly the hierarchy of DLEC constants for the well known five fundamental interactions (see the fifth column in the table 2). The eigenvalues of this matrix reveal the same hierarchy. This enables us to consider the interactions between “electric” fluxes  $\Phi_I^{(\text{el})}(\bar{m}, \underline{l})$  defined in (3.23), see also their interpretation after (3.31), as “pre-images” of the real fundamental interactions (or elementary pre-interactions [1]). Then in accordance with table 2 we shall relate the matrix elements  $Q^{+IJ}(0)$  to strong (for  $I = 1$ ), electromagnetic ( $I = 2$ ), weak ( $I = 3$ ), gravitational ( $I = 4$ ), and cosmological ( $I = 5$ ) pre-interactions. In this sense the “electric” fluxes in the BF-model acquire the status of quantized pre-fields bearing these names, *e.g.*,  $\Phi^{(\text{strong})}(\bar{m}, \underline{l}) := \Phi_1^{(\text{el})}(\bar{m}, \underline{l})$ ,  $\Phi^{(\text{electromagnetic})}(\bar{m}, \underline{l}) := \Phi_2^{(\text{el})}(\bar{m}, \underline{l})$ , and so on.

It is natural to suppose that diagonal elements of the other rational intersection matrices  $Q^{+IJ}(t)$ ,  $t \in \overline{-4, -1}$  have hierarchy of the vacuum-level coupling constants of the fundamental interactions (pre-interactions) acting at earlier phases of cosmological evolution (which correspond to the space-time manifolds modeled by cobordisms  $M_D^+(t)$ ). Thus our model includes a certain unification scheme of pre-interactions. So the intersection matrix  $Q^{+IJ}(-1)$  has the rank 4 and hence it describes the stage of universe with four fundamental pre-interactions. This stage can be associated with higher density of vacuum energy under which the topological structure of the universe is reconstructed. But it would be too speculative to directly connect this “unification” with the electroweak unification theory, since in our model five pre-interactions (between “electric” fluxes) are replaced by rather different (at least in the sense of hierarchy) four pre-interactions.

With the same reservations one can relate the  $3 \times 3$  matrix  $Q^{+IJ}(-2)$  to grand unified theories (GUT) in ordinary gauge terms. The next  $2 \times 2$  matrix  $Q^{+IJ}(-3)$  may be associated with a supersymmetric unification including the gravitation, since out of five low-energy (for  $t = 0$ ) pre-interactions there survive only two of them which correspond to gravitational and cosmological pre-interactions. In this case the cobordism  $M_D^+(-3)$  should pertain to the Planck scales. Then the  $1 \times 1$  matrix (one rational number)  $Q^{+IJ}(-3)$  might belong to the sub-Planckian level where only one pre-interaction (pre-image of the cosmological one) remains. It is obvious that in order these interrelations might have some sense, we should first introduce metric structures on the graph cobordisms  $M_D^+(t)$ , then constructing over them field theories with local degrees of freedoms. But we do not pose such a vast problem in this paper.

Now, let us see why manifestations of the presence of the exceptional orbits in  $\mathbb{Z}$ -homology spheres (which are spatial sections of our cosmological model), could be unobservable by astronomical means. The idea is essentially the same as in the inflation theory: to show that the linear scales of the present-epoch  $\mathbb{Z}$ -homology sphere  $\Sigma^+(0)$ , are by many orders of magnitude larger than the characteristic size of the observable part of the universe ( $L_0 \sim 10^{28}\text{cm}$ ). To evaluate the universe scales we take the following presumption. Let the four-dimensional volume of universe  $M^+(t)$  be proportional to  $\det Q_{IJ}^+(t)$  while the “minimal volume” in this universe be proportional to  $\det Q^{+IJ}(t)$ . Then the universe volume  $V^+(t)$  expressed in terms of the “minimal volume” should be  $\det Q_{IJ}^+(t) / \det Q^{+IJ}(t) = (\det Q^{+IJ}(t))^{-2}$

which yields the expression for the linear size of the universe as

$$L^+(t) \simeq \sqrt[4]{V^+(t)} = 1/\sqrt{\det Q^{+IJ}(t)}.$$

Numerical estimates give the following results:

$$\left. \begin{aligned} L^+(-4) &\sim 1.5 \times 10^4 \\ L^+(-3) &\sim 1.5 \times 10^{11} \\ L^+(-2) &\sim 8.0 \times 10^{23} \\ L^+(-1) &\sim 7.2 \times 10^{48} \\ L^+(0) &\sim 6.4 \times 10^{97} \end{aligned} \right\}. \quad (4.15)$$

As it was mentioned above, the state of universe corresponding to  $t = -3$  may be associated with a supersymmetric unification which includes the gravitational pre-interaction. If linear scales of the universe in this state might be considered as Planckian ones ( $L_{Pl} \simeq 1.6 \times 10^{-33}$  cm), then the hierarchy (4.15) would be expressed in centimeters:

$$\left. \begin{aligned} L^+(-4) &\sim 1.6 \times 10^{-40} \text{ cm} \\ L^+(-3) &\sim 1.6 \times 10^{-33} \text{ cm (normalization)} \\ L^+(-2) &\sim 8.6 \times 10^{-21} \text{ cm} \\ L^+(-1) &\sim 7.7 \times 10^4 \text{ cm} \\ L^+(0) &\sim 6.8 \times 10^{53} \text{ cm} \end{aligned} \right\}. \quad (4.16)$$

These estimates give a plausible picture of expansion of the universe in the course of cosmological evolution. Four periods of moderate inflation take place,

$$1.6 \times 10^{-40} \rightarrow 1.6 \times 10^{-33} \rightarrow 8.6 \times 10^{-21} \rightarrow 7.7 \times 10^4 \rightarrow 6.8 \times 10^{53},$$

which correspond to the sequence of topology changes (4.14). The size of the universe after the last “inflation” ( $\sim 6.8 \times 10^{53}$  cm) occurs to be 25 orders of magnitude greater than the size of its part which is observed now by means of the most sophisticated astronomical devices. These evaluations coincide with those obtained in  $T_0$ -discrete cosmological model [48] except for the last one.

Consequently, all that we astronomically observe is a three-dimensional almost flat disk about  $10^{28}$  cm in diameter cut out of the  $\mathbb{Z}$ -homology sphere whose characteristic size amounts  $6.8 \times 10^{53}$  cm. But while astronomical

observations then have nothing to do with spacetime topology, the local experiments providing information about the hierarchy of fundamental interactions (in contrast to ordinary inflation models) tell in our model sufficiently much about non-trivial topological structure of the spacetime. BF systems on graph cobordisms hint that the hierarchy of physical interactions originates at the global level (the utmost topological generalization of the Mach principle), so that the background vacuum (excitations-free) coupling constants naturally occur to coincide with basic topological invariants (intersection matrices) of the spacetime manifold. It is clear that the pre-interactions between “electric” fluxes in the framework of Abelian BF-model considered in this paper, cannot comprehensively express specific characteristics of the real fundamental interactions. However our model (in spite of exotic structure of the spacetime manifold or even due to these exotica) heuristically circumscribes certain properties of Nature.

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## Appendix

In this appendix we present the rational intersection matrices  $Q^-(t)$  corresponding to the cobordisms  $M_D^-(t)$  obtained from the splice diagrams in figure 9 applying the  $k^-$ -operation:

$$Q^-(0) = \begin{pmatrix} \mathbf{1} & 3.6 \times 10^{-2} & 0 & 0 & 0 \\ 3.6 \times 10^{-2} & \mathbf{3.6 \times 10^{-2}} & 9.4 \times 10^{-8} & 0 & 0 \\ 0 & 9.4 \times 10^{-8} & \mathbf{1.9 \times 10^{-7}} & 2.1 \times 10^{-24} & 0 \\ 0 & 0 & 2.1 \times 10^{-24} & \mathbf{1.5 \times 10^{-23}} & 1.3 \times 10^{-68} \\ 0 & 0 & 0 & 1.3 \times 10^{-68} & \mathbf{1.1 \times 10^{-113}} \end{pmatrix},$$

$$\lambda_I^-(0) = \{1., 3.4 \times 10^{-2}, 1.9 \times 10^{-7}, 1.5 \times 10^{-23}, 8. \times 10^{-139}\};$$



$$Q^{-}(-1) = \begin{pmatrix} 8.3 \times 10^{-2} & 1.1 \times 10^{-4} & 0 & 0 \\ 1.1 \times 10^{-4} & 4.3 \times 10^{-4} & 5.5 \times 10^{-13} & 0 \\ 0 & 5.5 \times 10^{-13} & 3.8 \times 10^{-12} & 3.1 \times 10^{-35} \\ 0 & 0 & 3.1 \times 10^{-35} & 2.5 \times 10^{-58} \end{pmatrix},$$

$$\lambda_I^{-}(-1) = \{8.3 \times 10^{-2}, 4.3 \times 10^{-4}, 3.8 \times 10^{-12}, 9.8 \times 10^{-72}\};$$

$$Q^{-}(-2) = \begin{pmatrix} 3.0 \times 10^{-3} & 1.5 \times 10^{-7} & 0 \\ 1.5 \times 10^{-7} & 2.0 \times 10^{-6} & 1.5 \times 10^{-18} \\ 0 & 1.5 \times 10^{-18} & 1.2 \times 10^{-30} \end{pmatrix},$$

$$\lambda_I^{-}(-2) = \{3.0 \times 10^{-3}, 2.0 \times 10^{-6}, 8.0 \times 10^{-37}\};$$

$$Q^{-}(-3) = \begin{pmatrix} 2.2 \times 10^{-6} & 2.1 \times 10^{-12} \\ 2.1 \times 10^{-12} & 2.2 \times 10^{-17} \end{pmatrix},$$

$$\lambda_I^{-}(-3) = \{2.2 \times 10^{-6}, 2.0 \times 10^{-17}\};$$

$$Q^{-}(-4) = (4.5 \times 10^{-9}).$$

It is easy to note that both diagonal elements of the matrix  $Q^{-}(0)$  and its eigenvalues show another hierarchy then that of the DLEC constants of the fundamental interactions (compare the boldface numbers in this matrix with the “experimental” column of the table 2 and with the corresponding characteristics of the matrix  $Q^{+}(0)$ ). So the matrices  $Q^{-}(t)$  may be related to some other universe, not with our one.

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